Classical Motives

Pol van Hoften
UCL, 17137331
pol.hoften.17@ucl.ac.uk

June 3, 2019

1 Introduction

In the previous talk, we defined a version of pure motives using existing Weil cohomology theories (étale, crystalline, de Rham, Betti). In this talk, we will take a different approach, and build our category of motives from geometric objects. The basic building blocks of our category of motives will be smooth projective varieties (not necessarily connected!) over our base field k. Of course, the category of smooth projective varieties $\mathbf{Var}_k$ is *not* abelian (or even additive), so we will have to do some modifications before things work.

In particular, morphisms will not be given by morphisms $f : X \to Y$ but by correspondences $\alpha : X \vdash Y$. These correspondences are (roughly) algebraic cycles living on $X \times Y$ and can be composed using intersection theory (which will be reviewed in the next section). In particular, if $f : X \to Y$, then the graph of f will give us a correspondence $\Gamma_f : Y \vdash X$. Note the contravariance! This is because we are trying to construct a universal cohomology theory, and cohomology theories are contravariant.

Correspondences are not just interesting because they provide us with an additive structure, they often induce interesting maps on cohomology. For a number theoretical example, consider Hecke correspondences on modular curves or more general Shimura varieties.

2 A crash course in intersection theory

In this section we let X be a smooth projective variety over a field k. We will discuss intersection theory on X, following [EH16].

Definition 1. *Let $Z^j(X)$ denote the free abelian group generated by irreducible reduced closed subschemes of X of codimension j and let $Z(X) = \bigoplus_j Z^j(X)$.*

We want to define an intersection product of cycles, but in order to do this we need to be able to 'move' cycles. Indeed, we would like to be able to make sense of the self intersection of cycles, which has no easy naive definition. In order to fix this, we will relate cycles by 'algebraic homotopies' parametrized by the 'unit interval' $\mathbb{P}^1$.

Definition 2. *Let $\mathrm{Rat}(X) \subset Z(X)$ be the subgroup generated by differences of the form*

$$[\Phi \cap \{t_0\} \times X] - [\Phi \cap \{t_1\} \times X]$$

where $t_0, t_1 \in \mathbb{P}^1$ and Φ is a subvariety of $\mathbb{P}^1 \times X$ not contained in any fiber $\{x\} \times X$. We say that two cycles are rationally equivalent if there difference is in $\operatorname{Rat}(X)$ and we define $\operatorname{CH}(X) = Z(X)/\operatorname{Rat}(X)$.

Theorem 1 (Theorem 1.5 [EH16])**.** *The Chow ring $\operatorname{CH}(X)$ is graded by codimension and there is a unique product structure on $\operatorname{CH}(X)$ such that if $A, B \subset X$ are closed subvarieties intersecting 'generically transverse', then*

$$[A] \cdot [B] = [A \cap B].$$

2.1 Functoriality

Let $f : X \to Y$ be a morphism of smooth projective varieties, then we expect there to be induced maps on Chow groups. There is a pullback map $f^* : \operatorname{CH}(Y) \to \operatorname{CH}(X)$, which is a graded ring homomorphism. It sends $[A]$ to $[f^{-1}(A)]$ if A is generically transverse to f and this probably determines it uniquely.

Furthermore, there is a pushforward map $f_* : \operatorname{CH}(X) \to \operatorname{CH}(Y)$ defined by

$$f_*[A] := \begin{cases} 0 & \text{if } \dim f(A) < \dim A \\ n.[f(A)] & \text{if } f\big|_A \text{ has degree } n \end{cases} .$$

Pushforward and pullback are related by the push-pull formula

$$f_*(f^*\alpha \cdot \beta) = \alpha \cdot f_*\beta.$$

We remark that pullback preserves the codimension of a cycle, while pushforward preserves the dimension of a cycle.

2.2 Equivalence relations on cycles

So far we have defined the ideal $\operatorname{Rat}(X) \subset Z(X)$ of cycles rationally equivalent to zero, but we will also need stronger equivalence relations. In particular, the Chow groups (with rational equivalence) can be infinitely generated (is this ever a problem?).

Definition 3. *Let H^* be a Weil cohomology theory. Then the we define an ideal of cycles homologically equivalent to zero as the kernel of the cycle map (this is given as part of the data of a Weil cohomology theory, c.f. last week's lecture).*

$$Z_{\text{Hom}}(X) := \ker\left(\operatorname{CH}(X) \otimes \mathbb{Q} \to H^*(X)\right).$$

Really, we are being sloppy with the notation here. From now on we only work with rational chow groups and we redefine

$$\operatorname{CH}(X) := \operatorname{CH}(X) \otimes \mathbb{Q}.$$

The intersection pairing on cycles is *not* a nondegenerate pairing, and so one might want to quotient out the kernel.

Definition 4. *Let $g \in \operatorname{CH}^j(X)$, then we call it numerically equivalent to zero if $\langle f, g \rangle = 0$ for all $g \in \operatorname{CH}^{\dim X - j}(X)$.*

2.3 Correspondences

In this section we fix an equivalence relation $\sim$ as in the previous section and we let $A(X) = \bigoplus_j A^j(X)$ denote cycles modulo $\sim$.

Definition 5. *Let X, Y be smooth projective varieties and assume that X has connected components $X_1, \cdots, X_n$ of dimensions $x_m, \cdots, d_m$ respectively. Then the group of correspondences of degree r from X to Y is*

$$\mathrm{Corr}^r(X, Y) := \bigoplus_{i=1}^{m} A^{d_i+r}(X_i \times Y).$$

If $\alpha \in \mathrm{Corr}^r(X, Y)$ and $\beta \in \mathrm{Corr}^s_\sim(Y, Z)$ then we define (it might be a useful exercise to figure out why this recovers the usual composition of morphisms by intersections of their graphs)

$$\alpha \circ \beta =: p_{13,*}\langle p_{12}^*\alpha, p_{23}^*\beta\rangle \in \mathrm{Corr}^{r+s}(X, Z),$$

where the morphisms p_{12}, p_{13}, p_{23} are the projection morphisms

$$\begin{array}{ccccc} & & X \times Y \times Z & & \\ & \swarrow^{p_{12}} & \downarrow^{p_{23}} & \searrow^{p_{13}} & \\ X \times Y & & Y \times Z & & X \times Z. \end{array}$$

In particular if $X = Y = Z$, then we get a ring structure on

$$\mathrm{Corr}(X, X) = A(X \times X)$$

which is not the ring structure coming from the intersection theory. In particular, it might be noncommutative (while the intersection pairing is not). Remark also that there is a subring $\mathrm{Corr}^0(X, X)$ of correspondences of degree 0.

If H^* is a Weil cohomology theory, then we remark that

$$\begin{aligned} H^*(X \times Y) &\cong H^*(X) \otimes H^*(Y) \\ &\cong H^*(X)^\vee \otimes H^*(Y) \\ &= \hom(H^*(X), H^*(Y)) \end{aligned}$$

using Poincaré duality and the Künneth formula. This means that the cycle class map

$$\mathrm{CH}(X \times Y) \to H^*(X \times Y)$$

gives an action of correspondences on cohomology. This is just a convoluted way of saying that correspondences act on cohomology by pullback-cup product-pushforward. In particular, the map

$$\mathrm{Corr}^0(X, X) \to \mathrm{End}\, H^*(X)$$

is a ring homomorphism for all X.

3 Definition

It is important to stress that the *construction* of the category of pure motives does not depend on any (standard) conjecture. We will discuss the (conjectural) properties of this category later. We will build the category of motives in three steps, starting from the category $\mathbf{Var}_k$ of smooth projective varieties over k. These steps are indicated in (1) below.

$$\mathbf{Var}_k \to C_\sim \mathbf{Var}_k \to \mathbf{Mot}^{\mathrm{eff}}_\sim(k) \hookrightarrow \mathbf{Mot}_\sim(k) \tag{1}$$

1. Let $C_\sim \mathbf{Var}_k$ be the category whose objects are smooth projective varieties and with $\hom(X, Y) = \mathrm{Corr}^0(X, Y)$. This category is a $\mathbb{Q}$-linear additive category, but is far from being abelian. In particular, idempotents p do not necessarily have images and kernels. For example, consider the cycle $\{x\} \times \mathbb{P}^1$ on $\mathbb{P}^1 \times \mathbb{P}^1$, then one checks that this is an idempotent correspondence. Moreover, it does not have an image since on cohomology it induces projection onto H^2.

   We can fix this in a 'universal' way, i.e., we can take a pseudo-abelian completion.

2. Let $\mathbf{Mot}^{\mathrm{eff}}_\sim(k)$ be the category whose objects are pairs

   $$(X, p)$$

   where X is a smooth projective variety and $p \in \mathrm{Corr}^0_\sim(X, X)$ is an idempotent correspondence. Now morphisms are given by

   $$\hom\left((X, p), (Y, q))\right) = q\,\mathrm{Corr}^0(X, Y)p;$$

   Equivalently, they are given by correspondences $\alpha : X \vdash Y$ such that $\alpha \circ p = \alpha$ and $q \circ \alpha = \alpha$, composition is given as above.

   If f is an idempotent endomorphism of (X, p), then one checks that

   $$(X, p) = (X, pfp) \oplus (X, p - pfp)$$

   gives us a decomposition of (X, p). Note that $pfp = f$ per definition.

3. Let $\mathbf{Mot}_\sim(k)$ be the category whose objects are triples

   $$(X, p, m)$$

   where (X, p) as before and $m \in \mathbb{Z}$ (which should be thought of as a Tate twist).

   $$\hom\left((X, p, m), (Y, q, n))\right) = q\,\mathrm{Corr}^{n-m}(X, Y)p;$$

   equivalently again as those $\alpha : X \vdash Y$ such that $\alpha \circ p = \alpha$ and $q \circ \alpha = \alpha$, composition is given as above.

Remark 1. The functor from $\mathbf{Var}_k \to \mathbf{Mot}_\sim(k)$ is given by

$$X \mapsto (X, 1, 0)$$

and $f : X \to Y$ is mapped to $\Gamma_f : (Y, id, 0) \vdash (X, id, 0)$.

Remark 2. There is a direct sum defined on objects by (at least if $m = n$, one has to work harder in general)

$$(X, p, m) \oplus (Y, q, m) := (X \cup Y, p \oplus q, m).$$

Remark 3. There is a tensor product given by

$$(X,p,m)\otimes(Y,q,n) := (X\times Y, p\otimes q, m+n).$$

What it does on morphisms is not hard to explain but we don't really have enough time. We define $\mathbf{1} := (\operatorname{Spec} k, id, 0)$ and $\mathbb{L} := (\operatorname{Spec} k, id, -1)$. We remark that $\mathbf{1}$ is the identity for the tensor product.

Remark 4. If X and Y are of pure dimension d, e respectively, then given a correspondence $\alpha \in \mathrm{Corr}^0(X,Y)$ we get an opposite correspondence ${}^t\alpha \in \mathrm{Corr}^{d-e}(Y,X)$ and so a morphism

$$\alpha : h(Y) \to h(X)\otimes \mathbb{L}^{e-d}.$$

This is really the reason we introduced the Lefschetz motive $\mathbb{L}$

Remark 5. If X is irreducible of $\dim d$ with a rational point $x \in X$, then we find that

$$\mathbf{1} \to h(X)$$

is a sub-object of $h(X)$, denoted by $h^0(X)$ and

$$h(x) \to \mathbb{L}^d$$

is a quotient object of d, denoted by $H^{2d}(X)$.

Remark 6. A cycle $f \in A^j(X)$ corresponds to a map

$$\mathbf{1} \to h(X)\otimes \mathbb{L}^j,$$

note that this is always a sub-object.

4 A universal Weil cohomology theory?

So far we have constructed a pseudo-abelian category $\mathbf{Mot}_\sim$ and now we 'would like to show' that it is a universal Weil cohomology theory. In particular our category should be semi-simple abelian and all Weil cohomology theories should factor through it. Given a Weil cohomology theory we would like to evaluate it on motives by

$$H^*(X,p,m) = pH^*(X)[2m],$$

that is, we take the image of the idempotent p and shift it up $2m$ degrees (since Weil cohomology theories take values in graded vector spaces). But what do we do with morphisms?

If $f : X \vdash Y$ is a correspondence, then it definitely induces a map on cohomology, as discussed before. However, we should get a map

$$\mathrm{Corr}^0_\sim \to H^*(X\times Y),$$

and the axioms of Weil cohomology theories only guarantee this existence when $\sim$ is rational equivalence. In particular, it is not known that all numerically trivial cycles are in the kernel of this map. Grothendieck conjectured this fact and this is usually referred to as standard conjecture D.

Alright, so why don't we just work with rational equivalence? Because the category $\mathbf{Mot}_{\mathrm{rat}}$ is not an abelian category! (at least when k is not contained in the algebraic closure of a finite field). Scholl [Sch] constructs an example of a morphisms with no kernel, using a non-torsion point on an elliptic curve. In the case of finite fields, there is a folklore conjecture (cite Milne!)

We will see in the next section that $\mathbf{Mot}_{\mathrm{num}}$ is a semi-simple abelian category. However, without conjecture D we do not know that this is a universal Weil cohomology theory. Moreover, even with conjecture D we still don't know much about this category. We expect that

$$H^*(X) = \bigoplus_{i=0}^{2n} H^i(X)$$

and also the hard Lefschetz theorem to hold on the level of motives. This is only known for abelian varieties (I think).

5 On numerical motives

Some things can be proven without the standard conjectures, and in 1991 a paper appeared which proved some nice results with seemingly elementary methods.

Theorem 2 ([Jan92])**.** *Let $\sim$ be an adequate equivalence relation and $\mathbf{M}_k$ the category of motives with respect to $\sim$, then the following are equivalent:*

(a) The category $\mathbf{Mot}_{\sim}(k)$ is abelian and semi-simple.

(b) For each smooth projective variety X, the ring $\hom_{\mathbf{M}_k}(X,X)$ is a finite dimensional semi-simple $\mathbb{Q}$-algebra.

(c) The relation $\sim$ is numerical equivalence.

Proof.

- ($a \Rightarrow c$) Let $f \in A^d(X)$ be a nonzero algebraic cycle corresponding to a morphism

  $$\mathbf{1} \to X \otimes \mathbb{L}^d$$

  Since $\mathbf{1}$ is indecomposable, this morphism will have a section, which corresponds to a cycle

  $$g \in A^{\dim X - d}(X)$$

  such that $\langle f, g\rangle = 1$. This means that f is not numerically equivalent to zero, so the given equivalence relation is coarser than numerical equivalence, but numerical equivalence is the coarsest adequate equivalence relation.

- ($c \Rightarrow b$) The fact that Chow groups modulo numerical equivalence are finite dimensional $\mathbb{Q}$-vector spaces is true in general (Find citation), so it remains to prove the semi-simplicity statement.

  For a $\mathbb{Q}$-algebra A that is Artinian (e.g. a finite dimensional vector space), semi-simplicity is equivalent to having trivial Jacobson radical. For such rings, the Jacobson radical can also be described as the largest two-sided nilpotent ideal. Therefore, our goal is to show that $J(A)$ is zero where

  $$A = \mathrm{Corr}^0_{\sim}(X, X).$$

Let H^* be a Weil cohomology theory (such as étale cohomology) and let $\sim'$ denote homological equivalence for this Weil cohomology theory. Let $B^j(X)$ denote cycle class group w.r.t. to $\sim'$ and note that there is a surjection

$$B^j(X) \to A^j(X)$$

given by killing numerically trivial cycles (conjecturally, this is an isomorphism). Now the Lefschetz trace formula tells us that for $f, g \in B^{\dim X}(X \times X)$ we have

$$\langle f, {}^t g \rangle = \sum_{i=0}^{2 \dim X} (-1)^i \operatorname{Tr}(f \circ g | H^i(X)).$$

Now let $f \in J(A$. In order to show that it is zero, it suffices to show that $\langle f, {}^t g \rangle = 0$ for all g since we are working with numerical equivalence. Let

$$S : B = \operatorname{Corr}^0_{\sim'} \to A$$

be the surjection described above and note that $S(J(B)) = J(A)$. Clearly we have $S(J(B))$ a nilpotent two-sided ideal (since S surjective) and so it is contained in $J(A)$. But $J(A)$ is the smallest ideal such that $A/J(A)$ is semisimple and $B/J(B)$ is also semisimple.

This means that we can lift f to an element $f' \in J(B)$ and since $J(B)$ is a twosided nilpotent ideal we find that $f' \circ g$ is nilpotent for all g. This also implies that the image of $f' \circ g$ is nilpotent in (the action on cohomology!)

$$\operatorname{End} H^i(X)$$

and so it has trace zero! We conclude that

$$\langle f, {}^t g \rangle = 0$$

which implies that $f' = 0$ and so $f = 0$ which shows that $J(A) = 0$ and A semi-simple.

- ($b \Rightarrow a$) This is true more generality, so we will formulate it as a lemma. The fact that the hypothesis of the Lemma hold for $\mathbf{C} = \mathbf{Mot}_\sim(k)$ is not obvious. It follows from the fact that

  $$\hom((X, p, m), (X, p, m)) = p \operatorname{Corr}^0(X, X) p$$

  and the fact that aRa is again a semi-simple ring of R semi-simple and a idempotent (is this even true?)

  Lemma 1 (Lemma 2 [Jan92]). *Let* $\mathbf{C}$ *be a* $\mathbb{Q}$*-linear, pseudo-abelian category such that* $End_{\mathbf{C}}(M)$ *is a finite-dimensional, semi-simple* $\mathbb{Q}$*-algebra for every object* M *of* $\mathbf{C}$*. Then* $\mathbf{C}$ *is a semi-simple abelian category.*

□

Proof of Lemma 1. Since $\operatorname{End}(M)$ is a finite-dimensional and semi-simple $\mathbb{Q}$-algebra, it is isomorphic to a finite product of matrix algebras over skew-fields over $\mathbb{Q}$. This means that indecomposable objects are precisely the ones whose endomorphism algebra is a skewfield over $\mathbb{Q}$. Moreover, every object is a finite direct sum of indecomposables (because idempotents have images and kernels in pseudo-abelian categories, and Wedderburn's theorem gives us lots of idempotents).

To show that the category is abelian, we have to show that two indecomposable objects M, N are either isomorphic or $\hom_{\mathbf{C}}(M, N) = 0$. This means that any map between objects of our category will be build up of endomorphisms of indecomposable objects (which will be simple) and zero maps, and that shows that they have kernels and cokernels.

So let M, N be decomposable and assume that $\hom_{\mathbf{C}}(M, N) \neq 0$. Then we claim that the composition

$$\hom_{\mathbf{C}}(N, M) \times \hom_{\mathbf{C}}(M, N) \to \mathrm{End}_{\mathbf{C}}(M)$$

is nonzero. Granting this claim for now, take (f, g) such that $f \circ g \neq 0$, then since $\mathrm{End}_{\mathbf{C}}(M)$ is a skewfield, the element $f \circ g$ is invertible and hence $(fg)^{-1} \circ f$ is an inverse of g. This means that $g : M \to N$ is a monomorphism and since N is indecomposable it is an isomorphism.

Suppose that the claim is false, then

$$\begin{pmatrix} 0 & 0 \\ \hom_{\mathbf{C}}(M, N) & 0 \end{pmatrix} \subset \begin{pmatrix} \mathrm{End}_{\mathbf{C}}(M) & \hom_{\mathbf{C}}(N, M) \\ \hom_{\mathbf{C}}(M, N) & \mathrm{End}_{\mathbf{C}}(N) \end{pmatrix} = \mathrm{End}_{\mathbf{C}}(M \oplus N)$$

is a non-trivial nilpotent two-sided *ideal* (here we really need finite dimensionality!). But them it must be contained in the Jacobson radical, which is trivial. □

Corollary 1. *Let k be a finite field (or contained in the algebraic closure of a finite field). Then the category $\mathbf{Mot}_{num}$ is a semi-simple $\mathbb{Q}$-linear Tannakian category*

References

[Jan92] Uwe Jannsen. "Motives, numerical equivalence, and semi-simplicity". In: *Invent. Math.* 107.3 (1992), pp. 447–452. ISSN: 0020-9910. URL: `https://doi.org/10.1007/BF01231898`.

[EH16] David Eisenbud and Joe Harris. *3264 and all that—a second course in algebraic geometry.* Cambridge University Press, Cambridge, 2016, pp. xiv+616. ISBN: 978-1-107-60272-4; 978-1-107-01708-5. URL: `https://doi.org/10.1017/CBO9781139062046`.

[Sch] Tony Scholl. *Classical motives.* URL: `https://www.dpmms.cam.ac.uk/~ajs1005/preprints/classical_motives.pdf`.