DERIVED STRUCTURES IN THE LANGLANDS PROGRAM - INTRODUCTION I

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A talk in the Derived Structures in the Langlands Program study group at UCL in Spring 2019. These are notes taken by Ashwin Iyengar (ashwin.iyengar@kcl.ac.uk) and edited by James Newton (j.newton@kcl.ac.uk). All errors/omissions/misrepresentations are James's responsibility.

1. Cohomology of Arithmetic Groups and Automorphic Forms

1.1. **Symmetric Spaces.** Let G be a semisimple algebraic group defined over $\mathbf{Q}$: as an example, one could take a number field $F/\mathbf{Q}$ and then take $G = \operatorname{Res}_{F/\mathbf{Q}} \operatorname{SL}_n$.

Then the **symmetric space** for G is given by $X = G(\mathbf{R})/K_\infty^\circ$, where K_∞° is a maximal connected compact subgroup of $G(\mathbf{R})$. If $G = \operatorname{Res}_{F/\mathbf{Q}}(\operatorname{SL}_2)$ as above and F has signature (r, s) (i.e. F has r real embeddings and s pairs of complex embeddings), then

$$X = (\operatorname{SL}_2(\mathbf{R})/\operatorname{SO}_2(\mathbf{R}))^r \times (\operatorname{SL}_2(\mathbf{C})/\operatorname{SU}(2))^s.$$

Note that $\operatorname{SL}_2(\mathbf{R})/\operatorname{SO}_2(\mathbf{R})$ is isomorphic to hyperbolic 2-space $\mathcal{H}^2$ (i.e. the complex upper half plane with the hyperbolic metric) and $\operatorname{SL}_2(\mathbf{C})/\operatorname{SU}(2)$ is similarly hyperbolic 3-space $\mathcal{H}^3$.

The symmetric space X is a real manifold, but for dimension reasons may not have a complex structure: for example, if $F/\mathbf{Q}$ is an imaginary quadratic extension, then $X \cong \mathcal{H}^3$, which is a 3-dimensional real manifold.

1.2. **Locally Symmetric Spaces.** Let $K = \prod_p K_p \subset G(\mathbf{A}_\mathbf{Q}^\infty)$, where $\mathbf{A}_\mathbf{Q}^\infty$ is the ring of finite adèles (of $\mathbf{Q}$) and each $K_p \subset G(\mathbf{Q}_p)$ is a compact open subgroup. Then the **locally symmetric space** attached to K (and G) is

$$Y(K) := G(\mathbf{Q})\backslash[X \times G(\mathbf{A_F})/K]$$

One can show that in fact, there exist finitely many arithmetic groups Γ_i acting on X for which

$$Y(K) = \bigsqcup_i \Gamma_i \backslash X$$

When Γ_i are small enough (neat) then each Γ_i acts freely and properly discontinously on X, and $Y(K)$ is naturally a smooth manifold (locally isomorphic to X, hence a *locally* symmetric space).

1.3. **Cohomology.** For us, the primary object of study will be the singular cohomology $H^*(Y(K), \mathbf{Z})$ which is alternatively computed as the group cohomology $\bigoplus_{i,n\geq 0} H^n(\Gamma_i, \mathbf{Z})$. There is a Hecke algebra $\mathbf{T}$ acting on $H^*(Y(K), \mathbf{Z})$, and the complex vector space $H^*(Y(K), \mathbf{C})$, together with its $\mathbf{T}$-module structure, can be described in terms of automorphic representations of G.

Example 1.3.1. If $G = \operatorname{SL}_{2,\mathbf{Q}}$, then this is the relationship between modular forms and cohomology of modular curves via the Eichler-Shimura isomorphism. In general, this relationship is given by a theorem of Franke (or Matsushima's formula if $Y(K)$ is compact, or we restrict attention to the contribution of *cuspidal* automorphic representations).

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Remark 1.3.1. Only a special subset of automorphic representations contribute to $H^*(Y(K), \mathbf{C})$: for cuspidal automorphic representations π we see a contribution to $H^*(Y(K), \mathbf{C})$ if π_∞ has non-vanishing $(\mathfrak{g}, K_\infty)$-cohomology (c.f. Matsushima's formula) and $(\pi^\infty)^K \neq 0$.

Fix a *cuspidal tempered* system of Hecke eigenvalues $\chi : \mathbf{T} \to \mathbf{Z}$. Here we say that a system of Hecke eigenvalues is cuspidal and tempered if all of the automorphic representations π with this system of Hecke eigenvalues are cuspidal and tempered (at Archimedean places).

Proposition 1.3.1 (Borel–Wallach + Matsushima's Formula)**.** *The generalized χ-eigenspaces $H^i(Y(K), \mathbf{Q})_\chi$ satisfy:*

(1) $H^i(Y(K), \mathbf{Q})_\chi = 0$ *if* $i \notin [q_0(G), q_0(G) + \ell_0(G)]$

(2) $\dim H^i(Y(K), \mathbf{Q})_\chi = \dim H^{q_0(G)}(Y(K), \mathbf{Q})_\chi \times \binom{\ell_0(G)}{i - q_0(G)}$

The integers $\ell_0(G)$ and $q_0(G)$ (especially $\ell_0(G)$) will be extremely important for the rest of the study group: we define

$$\ell_0(G) = \operatorname{rank} G(\mathbf{R}) - \operatorname{rank} K_\infty,$$

and we define

$$q_0(G) = \frac{1}{2}(\dim Y(K) - \ell_0(G)).$$

Example 1.3.2. Say $F/\mathbf{Q}$ has signature (r, s), as before. Let $G = \operatorname{Res}_{F/\mathbf{Q}} \mathrm{SL}_{2,F}$. Then $\ell_0(G) = s$ and $q_0(G) = r + s$. In this case

$$X = (\mathcal{H}^2)^r \times (\mathcal{H}^3)^s,$$

and $\dim Y(K) = 2r + 3s$. Therefore, $H^i(Y(K), \mathbf{Q})_\chi$ is nonzero for $i = r + s, \ldots, r + 2s$.

Note that $\binom{\ell_0(G)}{i - q_0(G)}$ is the dimension of $\bigwedge^{i - q_0(G)} V$ where V is a vector space of dimension $\ell_0(G)$. Thus, we are tempted by 1.3.1(2) to guess that $H^*(Y(K), \mathbf{Q})_\chi$ is generated by $H^{q_0(G)}(Y(K), \mathbf{Q})_\chi$ by the action of some exterior algebra acting on cohomology. In fact, this is exactly what Ventakatesh hopes to be true:

Conjecture 1.3.1 (Venkatesh, see for example the introduction to [7])**.** *$H^*(Y(K), \mathbf{Q})_\chi$ is generated by $H^{q_0(G)}(Y(K), \mathbf{Q})_\chi$ by the action of some exterior algebra acting on cohomology. In particular, this exterior algebra should come from a motivic cohomology group.*

1.3.1. *Addendum: The case of* $\operatorname{Res}_{F/\mathbf{Q}} \mathrm{GL}_1$*.* It's instructive to consider the example $G = \operatorname{Res}_{F/\mathbf{Q}} \mathrm{GL}_1$, F with signature (r, s). Of course this isn't semisimple, but we can set up everything discussed above for reductive groups as well. The analogue of the symmetric space is:

$$X := \mathrm{GL}_1(F \otimes_{\mathbf{Q}} \mathbf{R}) / \mathbf{R}_{>0} K_\infty^\circ$$

where $K_\infty^\circ = \mathrm{SU}(1)^s$ is the maximal connected compact subgroup of $G(\mathbf{R})$.

In this case, for $K \subset \mathrm{GL}_1(\mathbf{A}_F^\infty)$ compact open, we have a surjective map

$$Y(K) := F^\times \backslash [X \times \mathrm{GL}_1(\mathbf{A}_F^\infty)/K] \to Cl(K) := F^\times \backslash [\mathrm{GL}_1(\mathbf{A}_F^\infty) / \left((\mathbf{R}_{>0})^r \times (\mathbf{C}^\times)^s\right) K]$$

to a (*finite*) adelic generalized class group (if $K = \widehat{\mathcal{O}}_F$ then $Cl(K)$ is the narrow class group of F). Assuming that $F^\times \cap K$ is sufficiently small (more precisely, that $F^\times \cap K$ is a torsion-free finite index subgroup of the totally positive global units $\mathcal{O}_F^{\times,+}$), Dirichlet's unit theorem implies that the fibres of this map can be identified with a real torus of dimension $r + s - 1$:

$$T(K) = (F^\times \cap K) \backslash \left((\mathbf{R}_{>0})^r \times (\mathbf{C}^\times / SU(1))^s\right) / \mathbf{R}_{>0}.$$

In particular, the cohomology of $Y(K)$ is a direct sum of $|Cl(K)|$ copies of an exterior algebras on the free rank $r + s - 1$ Abelian group $F^\times \cap K$.

2. $\ell_0(G)$ and Galois Cohomology

Let $G = \mathrm{Res}_{F/\mathbf{Q}} \mathrm{SL}_{n,F}$, and let $\chi : \mathbf{T} \to \mathbf{Z}$ be a cuspidal tempered system of Hecke eigenvalues. Assume that $H^{q_0(G)}(Y(K), \mathbf{Q})_\chi \neq 0$.

We assume the following (vaguely stated) conjecture about the existence of Galois representations:

Conjecture 2.0.1. *For each prime p there exists a geometric Galois representation $\rho_\chi : G_F = \mathrm{Gal}(\overline{F}/F) \to \mathrm{PGL}_n(\overline{\mathbf{Q}}_p)$ such that $\rho_\chi(\mathrm{Frob}_v)$ is described in terms of χ for almost all places v of F.*

When F is CM and p is sufficiently large (depending on n), this conjecture is known (modulo the difference between SL_n and GL_n, which is not so serious) [5, 6, 1].

We can define Galois cohomology groups for the adjoint representation $\mathrm{Ad}\,\rho_\chi$ (which is an n^2-1 dimensional representation of G_F) and a Bloch-Kato Selmer group

$$H^1_f(G_F, \mathrm{Ad}\,\rho_\chi) \subset H^1(G_F, \mathrm{Ad}\,\rho_\chi).$$

There is also a dual Selmer group

$$H^1_f(G_F, (\mathrm{Ad}\,\rho_\chi)^*(1)) \subset H^1(G_F, (\mathrm{Ad}\,\rho_\chi)^*(1)).$$

where (1) denotes a Tate twist by the cyclotomic character.

Fact 2.0.1 (Greenberg-Wiles)**.** *Assuming ρ_χ is irreducible and* odd *(odd says something about the image of complex conjugation under ρ_χ)*[1]

$$\ell_0(G) = \dim H^1_f(G_F, (\mathrm{Ad}\,\rho_\chi)^*(1)) - \dim H^1_f(G_F, \mathrm{Ad}\,\rho_\chi).$$

This fact follows from a computation using Tate global duality, which is also a key computation in the Taylor–Wiles method (and it's extension by Calegari and Geraghty to situations with $l_0(G) > 0$. We should also note that we expect $\dim H^1_f(G_F, \mathrm{Ad}\,\rho_\chi) = 0$ — this would be a consequence of the Bloch-Kato conjecture. See [3].

Thus we see the constant $\ell_0(G)$ defined before on the "automorphic side" appearing in the completely different "Galois" side.

3. Patching and $H_*(Y(K), \mathbf{Z}_p)$

Here we give a Galois theoretic explanation of Venkatesh's conjecture and the exterior algebra structures appearing in Proposition 1.3.1, via the obstructed Taylor-Wiles method.

Let $\chi : \mathbf{T} \to \mathbf{Z}$ be as before, and fix a prime p. Then we may look at the reduced system of eigenvalues $\overline{\chi} : \mathbf{T} \to \mathbf{F}_p$. This determines a maximal ideal $\mathfrak{m} \subset \mathbf{T}$, and then $\mathbf{T}_\mathfrak{m}$ is a local $\mathbf{Z}_p$-algebra which can be shown to act on $H_*(Y(K), \mathbf{Z}_p)_\mathfrak{m}$ (we have switched from cohomology to homology here for convenience, although they encode the same information).

Assuming Conjecture 2.0.1, we then have a Galois representation ρ_χ attached to χ, and we can look at its reduction mod p, which we will denote

$$\overline{\rho}_\mathfrak{m} = \overline{\rho}_\chi : G_F \to \mathrm{PGL}_n(\overline{\mathbf{F}}_p).$$

In optimal circumstances, the Calegari-Geraghty method of [2] allows us to describe $H_*(Y(K), \mathbf{Z})_\mathfrak{m}$ in a rather elaborate way, using the following auxiliary objects:

- A map of power series algebras over $\mathbf{Z}_p$, $S_\infty \to R_\infty$ such that $\dim R_\infty = \dim S_\infty - \ell_0(G)$. (This numerology arises from the same Galois cohomology calculation as Fact 2.0.1)

[1] These properties are expected to always hold for ρ_χ

- A free R_∞-module M_∞, and an isomorphism
$$R_\infty \otimes_{S_\infty} \mathbf{Z}_p \cong R_{\overline{\rho}_\chi},$$
where the map $S_\infty \to \mathbf{Z}_p$ is given by sending all of the power series variables to 0, and where $R_{\overline{\rho}_\chi}$ is a certain geometric deformation ring of $\overline{\rho}_\chi$.

In nice enough cases (and assuming enough conjectures), Calegari–Geraghty show that $R_{\overline{\rho}_\chi} \cong \mathbf{T}_\mathfrak{m}$ and that
$$H_{q_0(G)+i}(Y(K), \mathbf{Z}_p)_\mathfrak{m} = \mathrm{Tor}_i^{S_\infty}(M_\infty, \mathbf{Z}_p).$$
Note that since R_∞ acts on M_∞, we get a graded action of $\mathrm{Tor}_*^{S_\infty}(R_\infty, \mathbf{Z}_p)$ on $H_*(Y(K), \mathbf{Z}_p)$.

Example 3.0.1. To see how this relates to Ventakesh's conjecture, suppose $\mathbf{T}_\mathfrak{m} = \mathbf{Z}_p$ (so we have a Galois representation $\rho_\mathfrak{m}$ with $\mathbf{Z}_p$ coefficients lifting $\overline{\rho}_\mathfrak{m}$). In this case we can take $R_\infty = \mathbf{Z}_p$ as well, and $S_\infty = \mathbf{Z}_p[\![x_1, \ldots, x_{\ell_0(G)}]\!]$. Then
$$\mathrm{Tor}_*^{S_\infty}(R_\infty, \mathbf{Z}_p) = \mathrm{Tor}_*^{\mathbf{Z}_p[\![x_1,\ldots,x_{\ell_0(G)}]\!]}(\mathbf{Z}_p, \mathbf{Z}_p),$$
which is the exterior algebra of a free rank $\ell_0(G)$ $\mathbf{Z}_p$-module. See, example, Corollary 4.5.5 and the subsequent exercises in [8].

Thus, we get the conjectured graded action, which should be motivic in origin — indeed the free rank $\ell_0(G)$ $\mathbf{Z}_p$-module which appears can be identified with the Selmer group $H^1_f(G_F, (\mathrm{Ad}\,\rho_\mathfrak{m})^*(1))$.

In [4], Galatius and Venkatesh describe $\mathrm{Tor}_*^{S_\infty}(R_\infty, \mathbf{Z}_p)$ as the homotopy groups of a simplicial ring, which is the *derived* deformation ring of the Galois representation $\overline{\rho}_\chi$. This recovers the Tor-algebra in a canonical way. In [7] (assuming various hypotheses and conjectures), Venkatesh shows that the action of the Tor-algebra on homology is also canonical, using the derived Hecke algebra which we briefly introduce next.

4. Derived Hecke Algebra

In addition to the Galois-theoretic explanation of the exterior algebra action, Venkatesh also gives a Hecke-theoretic explanation. One of the goals of [7] is to upgrade the action of $\mathbf{T}$ on $H^*(Y(K), \mathbf{Z}_p)$ to an action of a graded algebra $\widetilde{\mathbf{T}}$, whose degree zero part is $\mathbf{T}$. This is the "derived Hecke algebra". In particular, the action is graded, and we want a surjection (perhaps only after inverting p)
$$\widetilde{\mathbf{T}} \otimes_\mathbf{T} H^{q_0(G)}(Y(K), \mathbf{Z}_p)_\mathfrak{m} \twoheadrightarrow H^*(Y(K), \mathbf{Z}_p)_\mathfrak{m}.$$

When $\mathbf{T}_\mathfrak{m} = \mathbf{Z}_p$, Venkatesh proves that
$$\widetilde{\mathbf{T}}_\mathfrak{m} = \wedge^* H^1_f(G_F, (\mathrm{Ad}\,\rho_\mathfrak{m})^*(1))^*,$$
and compares the action of the derived Hecke algebra with the action of $\mathrm{Tor}_*^{S_\infty}(R_\infty, \mathbf{Z}_p) = \wedge^* H^1_f(G_F, (\mathrm{Ad}\,\rho_\mathfrak{m})^*(1))$.

References

[1] Patrick B. Allen, Frank Calegari, Ana Caraiani, Toby Gee, David Helm, Bao V. Le Hung, James Newton, Peter Scholze, Richard Taylor, and Jack A. Thorne. Potential automorphy over CM fields. *arXiv e-prints*, page arXiv:1812.09999, December 2018.

[2] Frank Calegari and David Geraghty. Modularity lifting beyond the Taylor-Wiles method. *Invent. Math.*, 211(1):297–433, 2018.

[3] Frank Calegari, David Geraghty, and Michael Harris. Bloch-Kato conjectures for automorphic motives. `http://www.math.uchicago.edu/~fcale/papers/BK.pdf`.

[4] S. Galatius and A. Venkatesh. Derived Galois deformation rings. *Adv. Math.*, 327:470–623, 2018.

[5] Michael Harris, Kai-Wen Lan, Richard Taylor, and Jack Thorne. On the rigid cohomology of certain Shimura varieties. *Res. Math. Sci.*, 3:3:37, 2016.

[6] Peter Scholze. On torsion in the cohomology of locally symmetric varieties. *Ann. of Math. (2)*, 182(3):945–1066, 2015.

[7] Akshay Venkatesh. Derived Hecke algebra and cohomology of arithmetic groups. *arXiv e-prints*, page arXiv:1608.07234, August 2016.

[8] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.