DEFORMATIONS OF GALOIS REPRESENTATIONS

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A talk in the Derived Structures in the Langlands Program study group at UCL in Spring 2019. These are notes taken by Ashwin Iyengar (ashwin.iyengar@kcl.ac.uk), edited by Misja Steinmetz.

1. Introduction

Why would one be interested in studying the deformation theory of Galois representations?

(1) To understand the results in this study group.

(2) They are used to prove Fermat's Last Theorem and the Shimura-Taniyama conjecture.

(3) They are used in proofs of the Sato-Tate conjecture.

(4) They are used in proofs of Serre's conjecture.

1.1. **Notation.** Let's fix some notation. In this talk, $K/\mathbf{Q}_p$ is a finite extension with ring of integers $\mathscr{O}$, uniformizer ϖ, and residue field k (for example, if K was unramified, then $\mathscr{O} = W(k)$ is just the Witt vectors of k). Let G be a profinite group, and fix a continuous n-dimensional representation $\overline{\rho} : G \to \mathrm{GL}_n(k)$.

1.2. **Rough idea of what's to come.** Roughly, we want to study lifts of ρ to local $\mathscr{O}$-algebras A with residue field k, that is, $\rho : G \to \mathrm{GL}_n(A)$ such that $\rho \mod \mathfrak{m}_A \cong \overline{\rho}$. Then the universal deformation ring R can be thought of as a parameter space for all lifts of $\overline{\rho}$.

For example, in nice cases (when the deformation problem is "unobstructed"), we will have $R = \mathscr{O}[\![x_1, \dots, x_r]\!]$, and if we let $\varphi : R \to A$ be an $\mathscr{O}$-algebra homomorphism taking $x_i \mapsto m_i$ for some chosen $m_i \in \mathfrak{m}_A$, then we get a composition of maps

$$\begin{array}{ccc} G & \xrightarrow{\text{universal map}} & \mathrm{GL}_n(R) \\ & \overset{\rho}{\searrow} & \downarrow{\scriptstyle\varphi} \\ & & \mathrm{GL}_n(A), \end{array}$$

and maybe we can think of $m_1, \dots, m_r$ as the "coordinates" defining the lift ρ. The deformation ring R can then be thought of as a "parameter space" with "parameters" x_i and we can recover the lift ρ by specialising the parameters $x_i \mapsto m_i$ to the "coordinates" of the lift.

Some references for this talk are Mazur's original paper [2] and Böckle's notes in [1].

2. Deformations of Representations of Profinite Groups

2.1. **Deformation Functors.** Let $\mathcal{C}_{\mathscr{O}}$ denote the category of local Artinian $\mathscr{O}$-algebras A together with a fixed isomorphism $A/\mathfrak{m}_A \xrightarrow{\sim} k$, whose morphisms are local homomorphisms respecting the isomorphism with k. Let $\widehat{\mathcal{C}_{\mathscr{O}}}$ denote the same category, replacing "local Artinian" with "complete local Noetherian".

Date: January 16, 2019.

Remark 2.1.1. If $A \in \widehat{\mathcal{C}_\mathscr{O}}$, then $A = \varprojlim_n A/\mathfrak{m}_A^n$, and each $A/\mathfrak{m}_A^n \in \mathcal{C}_\mathscr{O}$, so we may think of $\widehat{\mathcal{C}_\mathscr{O}}$ as the "completion" of $\mathcal{C}_\mathscr{O}$ in some suitable sense.

Definition 2.1.1. The *deformation functor* $D_{\overline{\rho}} : \widehat{\mathcal{C}_\mathscr{O}} \to \mathsf{Set}$ is defined as

$$D_{\overline{\rho}}^{\square}(A) = \{(\rho, M, \iota)\}/\sim,$$

where

(1) M is a free A-module of rank n,

(2) $\rho : G \to \mathrm{GL}_A(M)$ is a continuous representation

(3) ι is an isomorphism $\iota : \rho \otimes_A k \cong \overline{\rho}$,

and two such triples are equivalent if they are isomorphic via some isomorphism respecting the reduction map ι.

Definition 2.1.2. The *framed deformation functor* $D_{\overline{\rho}}^{\square} : \widehat{\mathcal{C}_\mathscr{O}} \to \mathsf{Set}$ is defined as

$$D_{\overline{\rho}}^{\square}(A) = \{(\rho, M, \iota, \beta)\}/\sim,$$

where ρ and M and ι are as above, and β is a basis of M lifting the standard basis of k^n under ι.

Equivalently, by picking coordinates, one can define $D_{\overline{\rho}}^{\square}(A)$ to be the set of continuous representations $\rho : G \to \mathrm{GL}_n(A)$ which reduce to $\overline{\rho}$ under the map $\mathrm{GL}_n(A) \to \mathrm{GL}_n(A/\mathfrak{m}_A) \cong \mathrm{GL}_n(k)$, i.e.

$$D_{\overline{\rho}}^{\square}(A) := \{\rho : G \to \mathrm{GL}_n(A) \mid \rho \bmod \mathfrak{m}_A = \overline{\rho}\}.$$

Then

$$D_{\overline{\rho}}(A) = D_{\overline{\rho}}^{\square}(A)/(\text{conjugation by } \ker(\mathrm{GL}_n(A) \to \mathrm{GL}_n(k))$$

Remark 2.1.2. Both $D_{\overline{\rho}}^{\square}$ and $D_{\overline{\rho}}$ are "continuous functors", where a functor $F : \widehat{\mathcal{C}_\mathscr{O}} \to \mathsf{Set}$ is continuous if

$$F(A) = \varprojlim_n F(A/\mathfrak{m}_A^n).$$

Therefore, we may compute the deformation functors on $\mathcal{C}_\mathscr{O}$.

2.2. ***p*-finiteness.** We want to find representing objects for $D_{\overline{\rho}}$ and $D_{\overline{\rho}}^{\square}$ in $\widehat{\mathcal{C}_\mathscr{O}}$, but we can only do this if G is "not too big". We make this precise now.

Definition 2.2.1. A profinite group G satisfies the *finiteness condition* Φ_p if for all open subgroups $H \leq G$, the $\mathbf{F}_p$-vector space $\mathrm{Hom}_{\mathrm{cts}}(H, \mathbf{F}_p)$ is finite-dimensional, or, equivalently, if the maximal pro-p quotient of H is topologically finitely generated.

We will mainly be concerned with the following two primary examples.

Example 2.2.1.

(1) Fix a finite extension $L/\mathbf{Q}_\ell$ (with possibly $\ell = p$). Then open subgroups of $G_L := \mathrm{Gal}(\overline{L}/L)$ are G_M for finite extensions M/L. But by local class field theory

$$\dim_{\mathbf{F}_p} \mathrm{Hom}_{\mathrm{cts}}(G_M, \mathbf{F}_p) = \dim_{\mathbf{F}_p} \mathrm{Hom}(M^\times/(M^\times)^p, \mathbf{F}_p) < \infty,$$

so Galois groups of p-adic local fields satisfy Φ_p.

(2) If F is a number field and S is a finite set of places then let $F_S \subset \overline{F}$ be the maximal extension of F unramified outside S. Similar arguments then show that $G_{F,S} = \mathrm{Gal}(F_S/F)$ satisfies Φ_p.

3. Tangent Spaces and Representability

3.1. **Tangent Spaces.** We let $k[\epsilon] = k[x]/(x^2)$ be the ring of dual numbers.

Definition 3.1.1. The tangent space to $D_{\overline{\rho}}$ is

$$\mathfrak{t}_{D_{\overline{\rho}}} := D_{\overline{\rho}}(k[\epsilon]).$$

Similarly, the tangent space to $D^{\square}_{\overline{\rho}}$ is

$$\mathfrak{t}_{D^{\square}_{\overline{\rho}}} := D^{\square}_{\overline{\rho}}(k[\epsilon]).$$

Remark 3.1.1. We have natural multiplication maps $k[\epsilon] \xrightarrow{\cdot a} k[\epsilon]$ for $a \in k$ by sending $x + y\epsilon \mapsto x + ay\epsilon$ and an addition map $k[\epsilon] \times_k k[\epsilon] \to k[\epsilon]$. Let D be either of the two functors considered above. Simply by functoriality we get maps $D(k[\epsilon]) \xrightarrow{D(\cdot a)} D(k[\epsilon])$ and $D(k[\epsilon] \times_k k[\epsilon]) \to D(k[\epsilon])$. We always have a natural map $D(k[\epsilon] \times_k k[\epsilon]) \xrightarrow{\sim} D(k[\epsilon]) \times_{D(k)} D(k[\epsilon])$ and it turns out this is a bijection. This allows us to define an appropriate addition map $\mathfrak{t}_D \times \mathfrak{t}_D \to \mathfrak{t}_D$ which gives $\mathfrak{t}_D = D(k[\epsilon])$ a k-vector space structure.

Recall that we denote the adjoint representation $\mathrm{End}_k(\overline{\rho})$ (with G acting by conjugation) by $\mathrm{ad}\,\overline{\rho}$.

Lemma 3.1.1.

(1) $D_{\overline{\rho}}(k[\epsilon]) \cong H^1(G, \mathrm{ad}\,\overline{\rho})$.
(2) *If G satisfies Φ_p, then* $\dim_k D_{\overline{\rho}}(k[\epsilon]) < \infty$.
(3) $D^{\square}_{\overline{\rho}}(k[\epsilon]) \cong Z^1(G, \mathrm{ad}\,\overline{\rho})$.

Proof. For (1), if $V \in D_{\overline{\rho}}(k[\epsilon])$, then $V/\epsilon V \cong \overline{\rho}$ and $\epsilon V \cong \overline{\rho}$, so there is an exact sequence

$$0 \to \overline{\rho} \to V \to \overline{\rho} \to 0,$$

so V defines a class in $\mathrm{Ext}^1(\overline{\rho}, \overline{\rho}) = H^1(G, \mathrm{ad}\,\overline{\rho})$ and one can check that this gives the required isomorphism.

For (2), note $H = \ker \overline{\rho}$. Then by the inflation-restriction exact sequence, we get

$$0 \to H^1(G/H, \mathrm{ad}\,\overline{\rho}) \to H^1(G, \mathrm{ad}\,\overline{\rho}) \to H^1(H, \mathrm{ad}\,\overline{\rho})^{G/H}$$

but $H^1(G/H, \mathrm{ad}\,\overline{\rho})$ is finite-dimensional (as both G/H and $\mathrm{ad}\,\overline{\rho}$ are), and

$$H^1(H, \mathrm{ad}\,\overline{\rho})^{G/H} = (\mathrm{Hom}(H, \mathbf{F}_p) \otimes \mathrm{ad}\,\overline{\rho})^{G/H}$$

is finite-dimensional as well (by Φ_p) so in fact $H^1(G, \mathrm{ad}\,\overline{\rho})$ is finite-dimensional.

For (3), given some $\rho \in D^{\square}_{\overline{\rho}}(k[\epsilon])$, we can write

$$\rho(g) = \overline{\rho}(g) + \epsilon\phi(g)\overline{\rho}(g).$$

It turns out that ρ being a group homomorphism means that $\phi(g) \in Z^1(G, \mathrm{ad}\,\overline{\rho})$, and this gives the required isomorphism. □

3.2. **Representability.** Question: when are the functors $D_{\overline{\rho}}$ and $D^{\square}_{\overline{\rho}}$ representable?

In general, let $F : \mathcal{C}_{\mathscr{O}} \to \mathsf{Set}$ be a functor. We say that F is *representable* if there exists an $R \in \widehat{\mathcal{C}_{\mathscr{O}}}$ such that we have isomorphisms

$$\mathrm{Hom}_{\mathscr{O}}(R, A) \xrightarrow{\sim} F(A)$$

which are functorial in $A \in \mathcal{C}_{\mathscr{O}}$. (In fancy language, we should possibly call this *pro-representable*, but I don't think there is a risk of confusion for the sake of this talk.)

Now suppose F is representable. Then it satisfies some conditions:

(1) $|F(k)| = |\mathrm{Hom}_{\mathscr{O}}(R, k)| = 1$.

(2) $\dim_k \mathfrak{t}_F < \infty$

(3) If $A \to C \leftarrow B$ are morphisms in $\mathcal{C}_{\mathcal{O}}$, then the natural map

$$F(A \times_C B) \to F(A) \times_{F(C)} F(B)$$

is bijective.

Fact 3.2.1. *These conditions are also sufficient for F to be representable. In fact, we can give a necessary and sufficient refinement of (3) called Schlessinger's criterion.*

We can use this criterion to prove

Proposition 3.2.1. *Assume* $\mathrm{End}_{k[G]}(\overline{\rho}) = k$ *and G satisfies Φ_p. Then $D_{\overline{\rho}}$ is representable. Call the representing object $R_{\overline{\rho}} \in \widehat{\mathcal{C}_{\mathcal{O}}}$* the universal deformation ring.

We will note prove this, but we will prove:

Proposition 3.2.2. *Assume G satisfies Φ_p. Then $D_{\overline{\rho}}^{\square}$ is representable. Call the representing object $R_{\overline{\rho}}^{\square}$ the* universal framed deformation ring.

Proof. First, assume G is finite, and pick a presentation $G = \langle g_1, \dots, g_s \mid r_1(g_1, \dots, g_s), \dots, r_t(g_1, \dots, g_s) \rangle$ such that the relations do not contain inverses.

Define

$$R = \mathcal{O}[x_{i,j}^k : i, j = 1, \dots, n, k = 1, \dots, s]/I,$$

where each $x_{i,j}^k$ is thought of as the (i,j)-th entry in a matrix X^k, and I is generated by the entries of the matrices

$$r_\ell((X^1), \dots, (X^s)) - I$$

for $\ell = 1, \dots, t$. Now let $J = \ker(R \to k)$, where the map $R \to k$ is given by sending $x_{i,j}^k$ to the (i,j)-th entry of $\overline{\rho}(g_k)$.

Then for the finite case, we have $R_{\overline{\rho}}^{\square} = \varprojlim_m R/J^m$, and we get a universal deformation $\rho^{\square} : g_k \mapsto (X^k) \in \mathrm{GL}_n(R_{\overline{\rho}}^{\square})$.

Then if G is profinite with $G = \varprojlim_i G/H_i$, then we define

$$R_{\overline{\rho}}^{\square} = \varprojlim_i R_i^{\square},$$

which will be a representing object of $D_{\overline{\rho}}^{\square}$. We need to prove it is Noetherian.

Let $R := R_{\overline{\rho}}^{\square}$. We have a natural isomorphism

$$\begin{array}{ccc} D_{\overline{\rho}}^{\square}(k[\epsilon]) = \mathrm{Hom}_{\mathcal{O}}(R, k[\epsilon]) & \xrightarrow{\sim} & \mathrm{Hom}_k(\mathfrak{m}_R/(\mathfrak{m}_R^2, \varpi), k) \\ \cup & & \cup \\ \left\{ \begin{smallmatrix} a + x \mapsto a + f(x)\epsilon \\ \text{for } a \in \mathcal{O} \text{ and } x \in \mathfrak{m}_R \end{smallmatrix} \right\} & \longleftarrow & f. \end{array}$$

Then by Φ_p we know that the tangent space and hence its dual $\mathfrak{m}_R/(\mathfrak{m}_R^2, \varpi)$ is finite-dimensional. Then we can use a Nakayama-type argument and completeness to lift a basis of this space to a generating set of R (as an $\mathcal{O}$-algebra) and therefore R is Noetherian. $\square$

Remark 3.2.1. We now have a universal lifting $\rho^\square : G \to \mathrm{GL}_n(R^\square_{\overline{\rho}})$ such that

$$\begin{array}{ccc} \{\text{liftings } \rho : G \to \mathrm{GL}_n(A)\} & \xleftrightarrow{\sim} & \{\mathscr{O}-\text{homs } R^\square_{\overline{\rho}} \xrightarrow{\varphi} A\} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (G \xrightarrow{\rho^\square} \mathrm{GL}_n(R^\square_{\overline{\rho}}) \xrightarrow{\varphi} GL_n(A)) & \longleftarrow\!\shortmid & \varphi \end{array}$$

3.3. **Presentations of Deformation Rings.**

Theorem 3.3.1. *Let* $r = \dim_k Z^1(G, \mathrm{ad}\,\overline{\rho})$*. Then there exists an* $\mathscr{O}$*-algebra isomorphism*

$$\mathscr{O}[\![x_1, \dots, x_r]\!]/(f_1, \dots, f_s) \cong R^\square_{\overline{\rho}}$$

where $s \leq \dim_k H^2(G, \mathrm{ad}\,\overline{\rho})$.

Proof. This is just a very rough outline. Since $\dim_k \mathfrak{m}_R/(\mathfrak{m}_R^2, \varpi\mathscr{O}) = r < \infty$, we can lift a basis to $\mathfrak{m}_R$, so that we get a surjection $\varphi : \mathscr{O}[\![x_1, \dots, x_r]\!] \twoheadrightarrow R^\square_{\overline{\rho}}$. Then we can show that if $J = \ker\varphi$, then there is an injection

$$(J/(\varpi, x_1, \dots, x_r)J)^\vee \hookrightarrow H^2(G, \mathrm{ad}\,\overline{\rho}).$$

$\square$

4. Deformation Conditions

We want to be able to impose extra conditions on our liftings (e.g. only consider crystalline lifts) without affecting representability. Towards this goal we make the following definition.

Definition 4.0.1. A *deformation condition* on (framed) deformations of $\overline{\rho}$ to $\mathcal{C}_{\mathscr{O}}$ is a property Q satisfying

(1) $\overline{\rho}$ satisfies Q;

(2) Given a deformation $\rho : G \to \mathrm{GL}_n(A)$ and an $\mathscr{O}$-algebra hom. $\varphi : A \to B$, the representation $\varphi \circ \rho$ also has property Q;

(3) If we have a fiber product

$$\begin{array}{ccc} A \times_C B & \xrightarrow{\rho} & A \\ \big\downarrow{\scriptstyle q} & & \big\downarrow{\scriptstyle \alpha} \\ B & \xrightarrow{\beta} & C \end{array}$$

and a deformation $\rho : G \to \mathrm{GL}_n(A \times_C B)$, then ρ has property Q if and only if $p \circ \rho$ and $q \circ \rho$ have property Q.

Definition 4.0.2. Let Q be a deformation condition for $\overline{\rho}$. Define a functor $D^{(\square)}_{\overline{\rho}} : \mathcal{C}_{\mathscr{O}} \to \mathsf{Set}$ by setting

$$D^{(\square)}_{\overline{\rho},Q}(A) := \{\text{(framed) deformations of } \overline{\rho} \text{ to } A \text{ satisfying } Q\}.$$

Then by the properties above $D^{(\square)}_{\overline{\rho},Q} : \mathcal{C}_{\mathscr{O}} \to \mathsf{Set}$ is a subfunctor which is relatively representable.

Example 4.0.1. Let $\chi : G \to \mathscr{O}^\times$ be a continuous character such that $\chi \bmod \varpi = \det\overline{\rho}$. Then we can define a universal framed deformation ring $R^\square_{\overline{\rho},\chi}$ whose lifts have fixed determinant χ. If $\mathrm{End}_k(\overline{\rho}) = k$, we get a universal deformation ring $R_{\overline{\rho},\chi}$.

4.1. **T-framed deformations.** Let $|S| < \infty$ be a finite set of places of a number field F, and let $G_v \hookrightarrow G_{F,S}$ be the decomposition group at each $v \in S$, and suppose we have a framed deformation condition Q_v on the universal framed deformation functor for $\overline{\rho}|_{G_v}$ – note that in this generality the unframed deformation function for $\overline{\rho}|_{G_v}$ may not be representable. Now assume $\mathrm{End}_k(\overline{\rho}) = k$ so that $D_{\overline{\rho}}$ is representable. Fix $A \in \mathcal{C}_{\mathcal{O}}$.

Definition 4.1.1. If $T \subset S$ is a subset, then a *T-framed deformation* of $\overline{\rho}$ of type $(S, \{Q_v\}, \chi)$ is an equivalence class in the set of pairs $(\rho, \{\alpha_v\}_{v \in T})$ such that

(1) $\rho : G \to \mathrm{GL}_n(A)$ is a lift such that $\det \rho = \chi$ and $\rho|_{G_v} \in Q_v$ for all $v \in S$
(2) $\alpha_v \in \ker(\mathrm{GL}_n(A) \to \mathrm{GL}_n(k))$

where the equivalence relation is given by $(\rho, \{\alpha_v\}_{v \in T}) \sim (\beta\rho\beta^{-1}, \{\beta\alpha_v\}_{v \in T})$.

In fact, this is representable, and gives us a way to look at deformations of a residual satisfying conditions at some places, which reduces the size of the universal deformation ring of $\overline{\rho} : G_{F,S} \to \mathrm{GL}_n(k)$. This will be important for the patching arguments, where we will need R to be small enough so that it is actually isomorphic to a properly defined Hecke algebra $\mathbf{T}$.

References

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[2] B. Mazur. Deforming Galois representations. In *Galois groups over* **Q** *(Berkeley, CA, 1987)*, volume 16 of *Math. Sci. Res. Inst. Publ.*, pages 385–437. Springer, New York, 1989.