THE CALEGARI–GERAGHTY METHOD

TOBY GEE

Abstract. This is based (very closely indeed in some places) on [Tho], as well as [CG18]. I also took a few things from [Sta13] without citation.

1. Commutative algebra

1.1. **Projective dimension.** Recall that if M is an A-module, then a projective resolution of M is an exact sequence

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0.$$

If A is Noetherian and M is a finite A-module, then we can take the P_i to be finite free.

1.1.1. **Definition.** We say that M has *projective dimension* d if there is a projective resolution of M with $P_n = 0$ for $n > d$, and if $P_d \neq 0$ for any projective resolution of M.

In particular, $\mathrm{proj.dim} M = 0$ if and only if M is projective.

In the case that $(A, \mathfrak{m})$ is local, projective modules are free, and we have the following notion of a minimal resolution.

1.1.2. **Definition.** Let $(A, \mathfrak{m}, k)$ be a local ring, and let M be an A-module. A projective resolution

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

is *minimal* if all of the maps $P_i \otimes_A k \to P_{i-1} \otimes_A k$ are zero.

By Nakayama's lemma, minimality is equivalent to demanding that for each i, P_{i-1} maps onto a minimal set of generators for $\mathrm{coker}\,(P_i \to P_{i-1})$. It's easy to see that if M is finitely generated then it has a minimal projective resolution by finite free A-modules, and in fact this resolution is unique in an appropriate sense (e.g. up to non-unique isomorphism in the derived category, see e.g. [KT17, Lem. 2.3]).

1.1.3. **Lemma.** *If $(A, \mathfrak{m}, k)$ is local, and M is a nonzero finite A-module, then* $\mathrm{proj.dim} M$ *is the length of every minimal free resolution of M, and is equal to the smallest integer i such that* $\mathrm{Tor}^A_{i+1}(k, M) = 0$.

Proof. We can compute $\mathrm{Tor}^A_{i+1}(k, M)$ by taking the homology of the tensor product of a projective resolution of M with k; so if $i \geq \mathrm{proj.dim} M$, then certainly $\mathrm{Tor}^A_{i+1}(k, M) = 0$.

Suppose that

$$0 \to P_n \to \cdots \to P_0 \to M \to 0$$

is a free resolution of M, and let i be minimal such that $\mathrm{Tor}_{i+1}^A(k,M)=0$; then $n \geq$ proj.dim$M \geq i$. If the resolution is minimal, then by definition all the differentials in the complex

$$0 \to P_n \otimes_A k \to \cdots \to P_0 \otimes_A k \to M \otimes_A k \to 0$$

are zero, so that $\mathrm{Tor}_{i+1}^A(k,M) = P_{i+1} \otimes_A k$ is zero if and only if $P_{i+1}=0$ if and only if $i \geq n$, as required. □

1.2. **Depth.** Matsumura ([Mat89, §16]) says: "The notion of depth is not very geometric, and rather hard to grasp, but is an extremely important invariant." That makes me feel a bit better, at least.

Let A be a ring and let M be an A-module. An element $a \in A$ is *M-regular* if $ax \neq 0$ for all $0 \neq x \in M$. A sequence $a_1, \dots, a_n \in A$ is an *M-regular sequence* if:

(1) For each $1 \leq i \leq n$, a_i is $M/(a_1,\dots,a_{i-1})M$-regular, and
(2) $M/(a_1,\dots,a_n)M \neq 0$.

1.2.1. *Remark.* Note that this depends on the order of the sequence. One can however show that if A is Noetherian, M is a finite A-module, and $a_1,\dots,a_n$ is an M-regular sequence with $(a_1,\dots,a_n) \subset \mathrm{rad}(A)$, then any permutation of $a_1,\dots,a_n$ is an M-regular sequence.

1.2.2. **Definition.** Let A be a ring, let I be an ideal of A, and let M be a finite A-module such that $M \neq IM$. Then the *I-depth of M* is by definition the length of a maximal M-regular sequence in I.

If A is local with maximal ideal $\mathfrak{m}$, then by the *depth of M* we mean the $\mathfrak{m}$-depth.

We will in fact only use depth in the Noetherian local case, where it behaves well; if you drop either Noetherian or local, it can misbehave.

1.2.3. **Lemma.** *Let $(A,\mathfrak{m},k)$ be a Noetherian local ring. Let M be a nonzero finite A-module. Then* $\mathrm{depth}(M)$ *is equal to the smallest integer i such that* $\mathrm{Ext}_A^i(k,M)$ *is nonzero.*

Proof. Let $i(M)$ denote the smallest integer i such that $\mathrm{Ext}_A^i(k,M)$ is nonzero. We will see by induction that $i(M) < \infty$. Note firstly that for the base case of the induction, we have $\mathrm{depth}(M)=0$ if and only if every element of $\mathfrak{m}$ is a zerodivisor on M, if and only if $\mathfrak{m}$ is contained in the union of the set $\mathrm{Ass}(M)$ of associated primes of M, if and only if $\mathfrak{m} \in \mathrm{Ass}(M)$ (by prime avoidance), i.e. if and only if $i(M)=0$.

Hence if $\mathrm{depth}(M)$ or $i(M)$ is >0, then we may choose $x \in \mathfrak{m}$ such that

- x is a nonzerodivisor on M, and
- $\mathrm{depth}(M/xM) = \mathrm{depth}(M) - 1$.

Consider the long exact sequence of Ext-groups associated to the short exact sequence $0 \to M \to M \to M/xM \to 0$:

$$\begin{aligned} &0 \to \mathrm{Hom}_A(k,M) \to \mathrm{Hom}_A(k,M) \to \mathrm{Hom}_A(k,M/xM) \\ &\to \mathrm{Ext}_A^1(k,M) \to \mathrm{Ext}_A^1(k,M) \to \mathrm{Ext}_A^1(k,M/xM) \to \dots \end{aligned}$$

Since $x \in \mathfrak{m}$ all the maps $\mathrm{Ext}_A^i(k,M) \to \mathrm{Ext}_A^i(k,M)$ are zero, so it is clear that $i(M/xM) = i(M) - 1$. Induction on $\mathrm{depth}(M)$ finishes the proof. □

1.2.4. **Theorem** (The Auslander–Buchsbaum formula)**.** *If* $(A, \mathfrak{m})$ *is a Noetherian local ring, and* M *is a finite* A*-module of finite projective dimension, then*

$$\text{proj.dim}M = \text{depth}(A) - \text{depth}(M).$$

Proof. We use induction on proj.dim$M$. If proj.dim$M = 0$ then M is free, so the result is clear. Otherwise, we look at the start of a minimal free resolution:

$$0 \to N \to F \to M \to 0$$

where F is chosen to have minimal rank, and $N \neq 0$. We have proj.dim$N =$ proj.dim$M - 1$ by Lemma 1.1.3, so it is enough to show that depth $M =$ depth $N - 1$.

Suppose firstly that depth $N <$ depth A. Looking at the long exact sequence of $\text{Ext}^i(k, -)$ from the short exact sequence above, we see that if $i + 1 <$ depth A, then the map $\text{Ext}^i(k, N) \to \text{Ext}^{i+1}(k, M)$ is an isomorphism, and by Lemma 1.2.3 we get depth $M =$ depth $N - 1$

Now suppose that depth $N \geq$ depth A. By the inductive hypothesis, we have depth $N =$ depth A, and N is projective (and we also have depth $F =$ depth A, since F is free). Writing $d =$ depth A, we have

$$0 \to \text{Ext}^{d-1}(k, M) \to \text{Ext}^d(k, N) \to \text{Ext}^d(k, F),$$

and we also see that $\text{Ext}^i(k, M) = 0$ if $i < d-1$. We need to show that $\text{Ext}^{d-1}(k, M) \neq 0$, so it is enough to show that the map $\text{Ext}^d(k, N) \to \text{Ext}^d(k, F)$ vanishes; but this follows from minimality, because the map $N \to F$ vanishes modulo $\mathfrak{m}$. □

1.2.5. **Lemma.** *Let* $(A, \mathfrak{m})$ *be a Noetherian local ring, and let* M, N *be nonzero finite* A*-modules. Then* $\text{Ext}^i_A(N, M) = 0$ *if* $i < \text{depth}(M) - \dim(N)$*.*

Proof. If $\dim(N) = 0$ then we filter N by copies of k and conclude by Lemma 1.2.3. Otherwise we can filter N by submodules whose successive subquotients are of the form A/P with P prime, and using long exact sequences of Ext^i we reduce to the case $N = A/P$. Since $\dim(N) > 0$ we can choose $x \in \mathfrak{m} \setminus P$, and the result then follows from a consideration of the long exact sequence of Ext^i coming from the short exact sequence

$$0 \to N \to N \to N/xN \to 0;$$

indeed we have $\dim(N/xN) = \dim N - 1$, so by induction we have $\text{Ext}^i(N/xN, M) = 0$ if $i < \text{depth}(M) - \dim(N) + 1$, and so we have that x kills $\text{Ext}^i(N, M)$ if $i < \text{depth}(M) - \dim(N)$, and we are done by Nakayama. □

1.2.6. **Corollary.** *If* $(A, \mathfrak{m})$ *is a Noetherian local ring,* M *is a finite* A*-module, and* $P \in \text{Ass}(M)$*, then* $\text{depth}(M) \leq \dim(A/P)$*.*

Proof. We have $\text{Hom}_A(A/P, M) \neq 0$, so this is immediate from Lemma 1.2.5. □

1.2.7. **Corollary.** *Let* A *be a Noetherian local ring. If* N *is a finite* A*-module, and* $0 \neq M \subseteq N$*, then* $\text{depth}(N) \leq \dim(M)$*.*

Proof. Let P be an associated prime of M (and hence of N). Then by Corollary 1.2.6, we have

$$\text{depth}(N) \leq \dim A/P \leq \dim(M). \qquad \square$$

We end this section with a standard result about regular local rings.

1.2.8. **Lemma.** *If* $(A, \mathfrak{m})$ *is a regular Noetherian local ring, then* $\text{depth}(A) = \dim(A)$*.*

Proof. By Corollary 1.2.6, we need to show that there is a regular sequence in $\mathfrak{m}$ of length $\dim A$. We will only need to apply the lemma in the case that A is a power series ring over a DVR, in which case the existence of such a regular sequence is clear. In fact, if A is regular, then any minimal set of generators of $\mathfrak{m}$ is a regular sequence, see [Sta13, Tag 00NQ]. □

1.3. **The key lemma.** The main new piece of commutative algebra that makes the Calegari–Geraghty version of Taylor–Wiles patching work is the following lemma.

1.3.1. **Lemma.** *Let $l_0 \geq 0$ be an integer and let S be a Noetherian* regular *local ring of dimension $d \geq l_0$. Let P be a perfect complex of S-modules which is concentrated in degrees $0, \ldots, l_0$. Then $\dim(H^*(P)) \geq d - l_0$, and moreover, if equality occurs, then:*

1. *P has a unique non-zero cohomology group, namely $H^{l_0}(P)$, and*
2. *$H^{l_0}(P)$ has depth $d - l_0$ and has projective dimension l_0.*

Proof. Let $\delta^i : P^i \to P^{i+1}$ denote the differential and let $m \leq l_0$ denote the smallest integer such that $H^m(P) \neq 0$. Consider the complex:

$$P^0 \to P^1 \to \ldots \to P^m.$$

By assumption, this complex is exact until the final term, and thus it is a projective resolution of the S-module $K^m := P^m / \operatorname{im}(\delta^{m-1})$. It follows that the projective dimension of K^m is $\leq m$. On the other hand, we see that

$$H^m(P) = \ker(\delta^m)/\operatorname{im}(\delta^{m-1}) \subseteq K^m,$$

and thus

$$d - \dim(H^m(P)) \leq d - \operatorname{depth}(K^m) = \operatorname{proj.dim}(K^m) \leq m,$$

where the first inequality is Corollary 1.2.7, and the equality is the Auslander–Buchsbaum formula, together with Lemma 1.2.8. So we have $\dim H^m(P) \geq d-m \geq d - l_0$, as required.

Suppose that $\dim(H^*(P)) \leq d-l_0$. Then it follows from the argument above that the smallest m for which $H^m(P)$ is non-zero is $m = l_0$, that $\dim(H^{l_0}(P)) = d - l_0$, that P is a resolution of $H^{l_0}(P)$, and that $\operatorname{proj.dim}(H^{l_0}(P)) = l_0$, completing the argument. □

2. The abstract Calegari–Geraghty–Taylor–Wiles setting

We now explain the importance of Lemma 1.3.1. When we have carried out the patching argument, we will have the following data. Let $E/\mathbf{Q}_p$ be a finite extension with ring of integers $\mathcal{O}$ and residue field k (e.g. in [Tho], we have $\mathcal{O} = W(k)$). Let C_0 be a perfect complex of $\mathcal{O}$-modules, concentrated in degrees $[q_0, q_0+l_0]$, and suppose that R is a complete local Noetherian $\mathcal{O}$-algebra with a homomorphism

$$R \to \operatorname{End}_{D(\mathcal{O})}(C_0).$$

Let $S_\infty = \mathcal{O}[[S_1, \ldots, S_r]]$ for some $r \geq 0$. Then we assume that we have:

- A perfect complex C_∞ of S_∞-modules, concentrated in degrees $[q_0, q_0+l_0]$.
- A complete local $\mathcal{O}$-algebra R_∞, with a surjection $\mathcal{O}[[X_1, \ldots, X_g]] \twoheadrightarrow R_\infty$, where $r - g = l_0$; and a map $S_\infty \to R_\infty$ equipped with an isomorphism $R_\infty \otimes_{S_\infty} \mathcal{O} \cong R$.
- An S_∞-algebra homomorphism $R_\infty \to \operatorname{End}_{D(S_\infty)}(C_\infty)$.

- An isomorphism of complexes of $\mathcal{O}$-algebras $C_\infty \otimes_{S_\infty} \mathcal{O} \cong C_0$ such that we have a commutative diagram

$$\begin{array}{ccc} R_\infty & \longrightarrow & \mathrm{End}_{D(S_\infty)}(C_\infty) \\ \downarrow & & \downarrow {\scriptstyle -\otimes^{\mathbf{L}}_{S_\infty}\mathcal{O}} \\ R & \longrightarrow & \mathrm{End}_{D(\mathcal{O})}(C_0) \end{array}$$

Now, using that the action of S_∞ factors through R_∞, and the assumption that $r - g = l_0$, we see that

$$\dim_{S_\infty} H^*(C_\infty) = \dim_{R_\infty} H^*(C_\infty) \leq \dim R_\infty \leq \dim \mathcal{O}[[X_1, \dots, X_g]] = \dim S_\infty - l_0,$$

where the first equality follows from the fact that $R_\infty/\mathrm{Ann}_{R_\infty}(H^*(C_\infty))$ is a finite $S_\infty/\mathrm{Ann}_{S_\infty}(H^*(C_\infty))$-algebra, because $R_\infty/\mathrm{Ann}_{R_\infty}(H^*(C_\infty))$ injects into the finite S_∞-module $\mathrm{End}_{D(S_\infty)}(C_\infty)$.

We can therefore apply Lemma 1.3.1 (with $S = S_\infty$, $P = C_\infty$), and we deduce that $H^{q_0+l_0}(C_\infty)$ has projective dimension l_0 as an S_∞-module, and depth and dimension both equal to $\dim S_\infty - l_0$. Now we have

$$\begin{aligned} \dim S_\infty - l_0 = \mathrm{depth}_{S_\infty} H^{q_0+l_0}(C_\infty) &\leq \mathrm{depth}_{R_\infty} H^{q_0+l_0}(C_\infty) \\ &\leq \dim R_\infty \\ &\leq \dim \mathcal{O}[[X_1, \dots, X_g]] = \dim S_\infty - l_0, \end{aligned}$$

where the first inequality is by definition (and the S_∞-action factoring through R_∞), and the second inequality is Corollary 1.2.6. Equality must hold in all the inequalities, so since $\mathcal{O}[[X_1, \dots, X_g]]$ is an integral domain and $\dim R_\infty = \dim \mathcal{O}[[X_1, \dots, X_g]]$, we have $R_\infty = \mathcal{O}[[X_1, \dots, X_g]]$.

By Auslander–Buchsbaum we must have $\mathrm{proj.dim}_{R_\infty} H^{q_0+l_0}(C_\infty) = 0$, so $H^{q_0+l_0}(C_\infty)$ is a projective R_∞-module. (Note that there is a subtlety here: we need to know that $H^{q_0+l_0}(C_\infty)$ has finite projective dimension over R_∞ in order to be allowed to apply Auslander–Buchsbaum! However, we have just shown that R_∞ is regular, and in fact every finite module over a regular local ring has finite projective dimension, see [Sta13, Tag 00O7].)

Passing to the quotient we see that $H^{q_0+l_0}(C_0) = H^{q_0+l_0}(C_\infty) \otimes_{S_\infty} \mathcal{O}$ (note that this is a tensor product, with no Tor terms, because we are in top degree) is free over $R_\infty \otimes_{S_\infty} \mathcal{O} = R$; in the applications, this also shows that R equals the Hecke algebra acting on C_0. Moreover, as explained in [GV18, §13], since we have shown that C_∞ is quasi-isomorphic to $H^{q_0+l_0}(C_\infty)$, we have that C_0 is quasi-isomorphic to $H^{q_0+l_0}(C_\infty) \otimes^{\mathbf{L}}_{S_\infty} \mathcal{O}$, and in particular the cohomology groups of C_0 have a free action of $\mathrm{Tor}^{S_\infty}(R_\infty, \mathcal{O})$.

3. Construction of complexes

3.1. You might now worry that we have to do a lot of work in order to construct the complexes that we need. In fact, the construction is very close to being the same as the one from Fred's talk last week. In particular, there are no changes at all to the Galois deformation theory or the construction of Taylor–Wiles primes. (One confusing subtlety that I am ignoring here, following [Tho], is that Taylor–Wiles patching is better phrased in terms of homology rather than cohomology; and indeed in [GV18, §13] things are written for homology.)

I will simply remind you of the setup and assumptions that we make. Assume that $p \geq 5$ is unramified in the number field F, that $\overline{\rho} : G_F \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ has inverse cyclotomic determinant, is finite flat at all places dividing p and unramified at all places not dividing p, and $\overline{\rho}|_{G_{F(\zeta_p)}}$ is irreducible. Then we have a universal deformation ring $R_{\overline{\rho}}$ for deformations satisfying these same conditions.

We let $G = \mathrm{PGL}_2/F$, and we let $X_U = G(F)\backslash G(\mathbf{A}_F)/UU_\infty$ for some appropriate compact open subgroup U. We have a Hecke algebra $\mathbf{T}_U$ action on a complex $C(X_U, \mathcal{O})$, and we conjecture that for the non-Eisenstein maximal ideal $\mathfrak{m}$ of $\mathbf{T}_U$ corresponding to $\overline{\rho}$, there is a homomorphism $R_{\overline{\rho}} \to \mathbf{T}_{U,\mathfrak{m}}$ satisfying natural local-global compatibility hypotheses. Note that this is an assumption about the existence of appropriate Galois representations for all degrees of cohomology.

As well as this assumption, we need the crucial Calegari–Geraghty vanishing assumption, that the cohomology groups $H^\star(X_U, k)_\mathfrak{m}$ are concentrated in degrees $[q_0, q_0+l_0] = [r_1, r_1 + r_2]$, where F has r_1 real places and $2r_2$ complex places.

The output of our patching construction is then as above, with $R = R_{\overline{\rho}}$ and $C_0 = C(X_U, \mathcal{O})$. S_∞ comes from the action of the diamond operators at the Taylor–Wiles primes, and the S_∞-algebra homomorphism $R_\infty \to \mathrm{End}_{D(S_\infty)}(C_\infty)$ ultimately comes from the first of our assumptions, and the assumption that C_∞ is concentrated in degrees $[q_0, q_0 + l_0]$ of course comes from the cohomological vanishing assumption

With this setup, there are only a few things that have to be dealt with differently to the case $l_0 = 0$. The patching argument now has to patch complexes rather than just modules. The basic idea here is that by replacing complexes with minimal resolutions, one can make everything appropriately finite. See [KT17, §3], or [GN16] for an approach using Scholze's version of the patching argument with ultrafilters.

There is also the issue of comparing the cohomology at different level structures at the Taylor–Wiles primes, which was more straightforward in the $l_0 = 0$ case, when the cohomology groups are free over the diamond operators. This comes down to a local calculation in the Iwahori Hecke algebra. In the case of GL_2 this is straightforward, and a nice approach for GL_n is in [KT17, §5]. This isn't dealt with in detail in [GV18, §13], but is explained for general G in [Ven16, Lem. 6.6].

Note finally that in fact we hardly ever know that the assumptions that we're making literally hold, even if F is totally real; but for the purposes of proving modularity lifting theorems, there are many possible weakenings of the assumptions, and tricks, so it is still possible to prove some useful modularity lifting theorems without proving the full strength of the conjectures.

References

[CG18] Frank Calegari and David Geraghty, *Modularity lifting beyond the Taylor–Wiles method*, Inventiones mathematicae **211** (2018), no. 1, 297–433.

[GN16] Toby Gee and James Newton, *Patching and the completed homology of locally symmetric spaces*, preprint, September 2016.

[GV18] S. Galatius and A. Venkatesh, *Derived Galois deformation rings*, Adv. Math. **327** (2018), 470–623.

[KT17] Chandrashekhar B. Khare and Jack A. Thorne, *Potential automorphy and the Leopoldt conjecture*, Amer. J. Math. **139** (2017), no. 5, 1205–1273.

[Mat89] Hideyuki Matsumura, *Commutative ring theory*, second ed., Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1989, Translated from the Japanese by M. Reid.

[Sta13] The Stacks Project Authors, *Stacks Project*, `http://stacks.math.columbia.edu`, 2013.

[Tho] Jack Thorne, *Beyond the Taylor–Wiles method.*

[Ven16] Akshay Venkatesh, *Derived Hecke algebra and cohomology of arithmetic groups*, arXiv e-prints (2016), arXiv:1608.07234.

E-mail address: `toby.gee@imperial.ac.uk`

Department of Mathematics, Imperial College London