MODEL CATEGORIES

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A talk in the Derived Structures in the Langlands Program study group at UCL in Spring 2019.
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The point is to understand the model categorical language in [GV].

Notation: throughout, C will be a category. $Map(C)$ will be the "arrow category" of C, in which objects are morphisms, and morphisms are commutative squares.

Definition 0.1.

- A map $f \in Map(C)$ is a **retract** of a map g if we have a diagram

$$\begin{tikzcd}
B \arrow[r] \arrow[d, "f"] \arrow[rr, bend left, "\mathrm{id}"] & A \arrow[r] \arrow[d, "g"] & B \arrow[d, "f"] \\
B' \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & A' \arrow[r] & B',
\end{tikzcd}$$

- A **functorial factorization** is a ordered pair (α, β) of functors $Map(C) \to Map(C)$ such that $f = \beta(f) \circ \alpha(f)$ for all $f \in Map(C)$.
- If $i : A \to B$ and $p : X \to Y$ are maps in C then i has the **left lifting property (LLP)** with respect to p and p has the **right lifting property (RLP)** with respect to i if you have a lift h in the diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"] & X \arrow[d, "p"] \\
B \arrow[r] \arrow[ur, dashed, "h"] & Y
\end{tikzcd}$$

  for any maps $A \to X$ and $B \to Y$.

Definition 0.2. A **model structure** on C is a choice of three subcategories of $Map(C)$ called **weak equivalences**, **cofibrations**, and **fibrations** and two functorial factorizations (α, β) and (γ, δ) satisfying:

(1) 2-of-3 property: if $f, g \in Map(C)$ are such that $g \circ f$ defined and if any two of $f, g, g \circ f$ are weak equivalences, then so is the third.

(2) If f, g are morphisms and f is a retract of g and g is a weak equivalence (respectively fibration, cofibration), then so is f.

(3) We say that a weak equivalence which is also a cofibration (resp. fibration) is a **trivial cofibration** (resp. **trivial fibration**). Then trivial cofibrations have the LLP with respect to fibrations, and trivial fibrations have the RLP with respect to cofibrations.

(4) For any morphism $f \in Map(C)$, we have $\alpha(f)$ is a cofibration, $\beta(f)$ is a trivial fibration, $\gamma(f)$ is a trivial cofibration, and $\delta(f)$ is a fibration.

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Definition 0.3. A **model category** is a category C with all small limits and colimits together with a model structure.

Note that a model category C has all small limits and colimits, and therefore has an initial object $*$ and a terminal object $\bullet$.

Definition 0.4. We say that an object $b \in C$ is **cofibrant** if $* \to b$ is a cofibration, and **fibrant** if $b \to \bullet$ is a fibration.

Lemma 0.5. *Let C be a model category. A map f is a cofibration if and only if it has the LLP with respect to all trivial fibrations. Similarly, f is a fibration if and only if it has the right lifting property with respect to all trivial cofibrations.*

In practice, this means that a model structure only depends on the weak equivalences and either the fibrations, or cofibrations.

1. EXAMPLES

(1) (Trivial example) Let C be a category with small limits and colimits. Then set one of the distinguished classes to be isomorphisms, and the others to be all maps. This determines a model structure.

(2) Let Top denote the category of topological spaces. Define a model structure as follows:
   - The weak equivalences are weak homotopy equivalences (i.e. induce isomorphisms on homotopy groups),
   - The fibrations are Serre fibrations (i.e. maps with the RLP with respect to $D^n \to D^n \times I, x \mapsto (x, 0)$ for all n where D^n is a topological n-disk and I is an interval), and
   - The cofibrations are uniquely determined.

   Here all objects are fibrant, and the cofibrant objects are the CW complexes.

(3) Simplicial sets:
   - Weak equivalences are weak homotopy equivalences (i.e. morphisms whose geometric realizations are weak homotopy equivalences,
   - Fibrations are Kan fibrations, defined last week.
   - Cofibrations are monomorphisms in sSet, equivalently level-wise injective maps.

   Fibrant objects are Kan complexes (by definition), and all simplicial sets are cofibrant because the initial object in sSet is empty, levelwise.

(4) Chain complexes. Let R be a (not necessarily commutative) ring and $\mathrm{Ch}(R)$ the category of chain complexes (bounded above) of left R-modules.
   - The weak equivalences are quasi-isomorphisms of chain complexes (i.e. isomorphisms on homology groups),
   - $f : M \to N$ is a fibration if for all k, $f_k : M_k \to N_k$ is surjective, and
   - $f : M \to N$ is a cofibration if for all k, $f_k : M_k \to N_k$ is injective and the cokernel is a projective R-module.

   Here, all fibrant objects are fibrant, and cofibrant R-modules are exactly the projective R-modules.

Lemma 1.1 (Ken Brown's Lemma)**.** *Let C be a model category and D be a category with a chosen class of weak equivalences. Let $F : C \to D$ be a functor. Then*

- *If F sends trivial cofibrations between cofibrant objects to weak equivalences, then it sends all weak equivalences between cofibrant objects to weak equivalences.*
- *If F sends trivial fibrations between fibrant objects to weak equivalences, then it sends all weak equivalences between fibrant objects to weak equivalences.*

Proof Sketch. Let $f : A \to B$ be a weak equivalence between cofibrant objects. We can factor $(f, \mathrm{id}_B) : A \sqcup B \to B$ as

$$A \sqcup B \xrightarrow{q} C \xrightarrow{p} B$$

with q a cofibration and p a trivial fibration. Then using that cofibrations are closed under pushouts, we have from the diagram $* \to A$ $* \to B$ $B \xrightarrow{i_1} A \sqcup B$ $A \xrightarrow{i_2} A \sqcup B$ that i_1, i_2 are cofibrations, and by the 2-of-3 property, we get $f \circ i_1$ and $g \circ i_2$ are weak equivalences, and thus trivial cofibrations. So by hypothesis we have $F(f \circ i_1)$ and $F(f \circ i_2)$ are weak equivalences, and

$$F(p \circ q \circ i_2) = F(\mathrm{id}_B)$$

is also a weak equivalence, so $F(p)$ is a weak equivalence. Thus, $F(f) = F(p \circ q \circ i_1)$ is a weak equivalences. □

2. Homotopy Category

Definition 2.1. Let C be a category with a subset W of weak equivalences (i.e. satisfying 2-of-3 property). Then the homotopy category of (C, W) is defined as follows:

- First construct the free category $F(C, W^{-1})$, whose objects are objects of C, and whose morphisms are strings of composable morphisms in C and inverses of morphisms in W.
- Then take the quotient of $F(C, W^{-1})$ by the relations
  - $\mathrm{id}_A = (\mathrm{id}_A)$
  - $(f, g) = (g \circ f)$
  - If $w : c \to c'$ is a morphism, then $\mathrm{id}_c = (w, w^{-1})$ and $\mathrm{id}_{c'} = (w^{-1}, w)$.

Proposition 2.2. *Let C be a model category, and let C_c, C_f, C_{cf} denote the subcategories of cofibrant objects, fibrant objects, and fibrant-cofibrant (i.e. both) objects. Then the inclusions*

$$\mathrm{Ho}(C_{cf}) \to \mathrm{Ho}(C_c) \to \mathrm{Ho}(C)$$

and

$$\mathrm{Ho}(C_{cf}) \to \mathrm{Ho}(C_f) \to \mathrm{Ho}(C)$$

are equivalences of categories.

Recall that if C is a model category, we have functorial factorizations β, α such that α gives cofibrations and β gives trivial fibrations. We can use this factorization to to factor $* \to b$ for any $b \in C$ as

$$* \xrightarrow{\alpha(f)} Q(b) \xrightarrow{\beta(f)} b.$$

We call $Q(b)$ the **cofibrant replacement functor**. Dually, we can use (δ, γ) to get a **fibrant replacement functor** R.

Definition 2.3. Let C be a model category. Let $f, g : B \to X$ be two maps.

(1) A **cylinder object** B' of B is a factorization of $B \sqcup B \xrightarrow{(\mathrm{id}_B, \mathrm{id}_B)} B$ as

$$B \sqcup B \xrightarrow{i_1 \sqcup i_2} B' \xrightarrow{s} B$$

with (i_1, i_2) a cofibration and s a weak equivalence.

(2) A path object X' for X is a factorization of the diagonal map $X \xrightarrow{(\mathrm{id}_X \times \mathrm{id}_X)} X \times X$

$$X \to X' \xrightarrow{j_1 \times j_2} X \times X$$

with $X \to X'$ a weak equivalence and $X' \to X \times X$ a fibration.

(3) A left homotopy from f to g is a map $H : B' \to X$ (for B' a cylinder object of B) such that

$$Hi_1 = f \text{ and } Hi_2 = g.$$

We write this as $f \sim^{\ell} g$.

(4) A right homotopy from f to g is a map $H : B \to X'$ (for X' a path object for X) such that

$$j_1 H = f \text{ and } j_2 H = g.$$

We write this as $f \sim^{r} g$.

(5) If $f \sim^{\ell} g$ and $f \sim^{r} g$, then we say that $f \sim g$, i.e. f is homotopic to g.

(6) We say f is a homotopy equivalence if there exists $h : X \to B$ such that $hf \sim id_B$ and $fh \sim \mathrm{id}_X$.

Theorem 2.4 (Whitehead)**.** *Let C be a model category, and let $\gamma : C \to \mathrm{Ho}(C)$ denote the quotient map.*

(1) *A map of C_{cf} is a weak equivalence if and only if it is a homotopy equivalence. Moreover, there is a unique isomorphism between $C_{cf}/\sim$ (quotient by the relation of homotopy) and $\mathrm{Ho}(C_{cf})$.*

(2) *The inclusion $C_{cf} \to C$ induces an equivalence of categories*

$$C_{cf}/\sim \to \mathrm{Ho}(C).$$

(3) *There are natural isomorphisms*

$$C(Q(R(X)), Q(R(Y))/\sim \xrightarrow{\sim} \mathrm{Ho}(C)(\gamma(X), \gamma(Y)) \xrightarrow{\sim} C(R(Q(X)), R(Q(Y)))/\sim .$$

Moreover, there is a natural isomorphisms $\mathrm{Ho}(C)(\gamma(X), \gamma(Y)) \xrightarrow{\sim} C(Q(X), R(Y))/\sim$. If X is cofibrant and Y is fibrant, then there is a natural isomorphism

$$C(X, Y)/\sim \xrightarrow{\sim} \mathrm{Ho}(C)(\gamma(X), \gamma(Y)).$$

(4) *If $f : A \to B$ is a map such that $\gamma(f)$ is an isomorphism, then f must be a weak equivalence.*

3. QUILLEN FUNCTORS

Let C, D be model categories.

(1) A functor $F : C \to D$ is a **left Quillen functor** if F is a left adjoint and it preserves cofibrations and trivial cofibrations.

(2) A functor $U : D \to C$ is a **right Quillen functor** if it is a right adjoint and preserves fibrations and trivial fibrations.

(3) Note (F, U) is a **Quillen adjunction** if (F, U) if (F, U) is an adjoint pair and either F or U is a Quillen functor (these are equivalent).

(4) If F is a left Quillen functor, we define the **total left derived functor** $LF : \mathrm{Ho}(C) \to \mathrm{Ho}(D)$ as the composition

$$\mathrm{Ho}(C) \xrightarrow{\mathrm{Ho}(Q)} \mathrm{Ho}(C_c) \xrightarrow{\mathrm{Ho}(F)} \mathrm{Ho}(D).$$

Similarly, U defines a **total right derived functor** $RU : \mathrm{Ho}(D) \to \mathrm{Ho}(C)$ as the composition

$$\mathrm{Ho}(D) \xrightarrow{\mathrm{Ho}(R)} \mathrm{Ho}(D_f) \xrightarrow{\mathrm{Ho}(U)} \mathrm{Ho}(C).$$

Lemma 3.1. *If (F, U) is a Quillen adjunction, then (LF, RU) is an adjoint pair on the homotopy categories.*

As an example, recall the functor $X \mapsto \operatorname{Sing}(X)$ taking a topological space X to its singular simplicial set, and the functor $S \mapsto |S|$, the geometric realization. In fact, the pair $(|\cdot|, \operatorname{Sing}(\cdot))$ is a Quillen adjunction.

We can use Quillen adjunctions to transport model structures as follows. Let C be a model category, and let $(F : C \to D, U : D \to C)$ be an adjoint pair for some category D. Assuming U preserves filtered colimits, we can define a model structure on D from the one on C as follows.

1. A weak equivalence in D is a map whose image under U is a weak equivalence in C.
2. A fibration in D is a map whose image under U is a fibration in C.
3. Cofibrations are then determined by the LLP with respect to trivial fibrations.

This is Theorem 5.1 of [1].

REFERENCES

[1] Paul G. Goerss and John F. Jardine. *Simplicial homotopy theory*, volume 174 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1999.