Simplicial commutative rings

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notes by Pol van Hoften (all errors due to him and Ashwin Iyengar)

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1 Classical setup

Let k be a field and let $\mathbf{Art_k}$ be the category of Artin local rings with residue field k and let $\mathcal{F} : \mathbf{Art_k} \to \mathbf{Set}$ be a functor ("deformation problem"). We are interested in properties of these kinds of functors, for example (pro)-representability. Today we want to replace this classical setup with a derived setup. Replace sets with simplicial sets and $\mathbf{Art_k}$ with $\mathbf{sArt_k}$ and functors with simplicially enriched functors.

2 Simplicial commutative rings

Definition 1. *The category of simplicial commutative rings* $\mathbf{sCR}$ *is the category of simplicial objects in the category of commutative rings, i.e., the functor category*

$$[\Delta^{op}, \mathbf{CR}].$$

This is the same thing as ring objects in the category of simplicial sets (because limits are computed pointwise).

The free-forgetful adjunction

$$\text{Forget} : \mathbf{CR} \leftrightarrow \mathbf{Sets} : \mathbb{Z}[-]$$

extends to an adjunction

$$\mathbf{sCR} \leftrightarrow \mathbf{sSets}$$

by applying the polynomial ring functor to the set of n-simplices. We can use this adjunction to transfer the model structure from $\mathbf{sSets}$ to $\mathbf{sCR}$, which has the following description: A map $f : R \to S$ is

- Weak equivalence if and only if the map of the underlying simplicial sets is a weak equivalence.
- Fibration if and only if the map of the underlying simplicial sets is a (Kan) fibration.
- Cofibration if and only if it satisfies the left lifting property (LLP) with respect to trivial fibrations.

Remark 1. Every simplicial commutative ring is in particular a simplicial (abelian) group, and so it is fibrant.

2.1 Enrichment

Recall that **sSets** is self-enriched, i.e., it has internal hom objects (this is true just because it is a presheaf category). These have the explicit description

$$\begin{aligned}\underline{\mathrm{Hom}}(X,Y)_n &:= \hom(\Delta^n, \underline{\mathrm{Hom}}(X,Y)) \\ &= \hom(X \times \Delta^n, Y)\end{aligned}$$

and we use the notation $Y^X := \underline{\mathrm{Hom}}(X,Y)$ when it is convenient.

Fact 1. If $i : X \to Y$ is a cofibration and $p : A \to B$ is a fibration (of simplicial sets), then the induced map

$$A^Y \to A^X \times_{B^X} B^Y$$

is a fibration and it is a trivial fibration if either i or p is trivial.

For simplicial commutative rings R, S, we can form the equalizer

$$\underline{\mathrm{Hom}}(R,S) \underset{\mathrm{ev}(0_R)}{\overset{0}{\rightrightarrows}} S$$

which is the subobject of "0-preserving maps". Similarly we can define the subobject of maps "Preserving 1", that are "additive", "multiplicative" and taking the intersection we get an object

$$\underline{\mathbf{sCR}}(R,S).$$

For $n \geq 0$ the mapping complex S^{Δ^n} has the structure of a simplicial ring and we can describe

$$\underline{\mathbf{sCR}}(R,S)_n = \mathbf{sCR}(R, S^{\Delta^n}).$$

This gives the category **sCR** the structure of a category enriched over **sSets**.

3 Simplicial Artin local rings

Write $I = \Delta^1$ and we define the boundary of the n-cube by

$$\partial I^n = \bigcup_{1 \leq k \leq n} I^{k-1} \times \partial I \times I^{n-k}.$$

The simplicial circle S^n is then defined to be the pushout (so it is naturally a pointed simplicial set)

$$\begin{array}{ccc} \partial I^n & \longrightarrow & I^n \\ \downarrow & & \downarrow \\ \{*\} & \longrightarrow & S^n \end{array}$$

This is not the usual definition but it has the advantage that

$$S^{n+m} := S^n \wedge S^m := (S^n \times S^m) / S^n \times \{*\} \cup \{*\} \times S^n$$

holds on the nose, rather than up to homotopy. For a simplicial commutative ring R we define

$$\begin{aligned}\pi_n(R) &:= \hom((S^n, \{*\}), (R, 0))/\sim \\ &= \pi_0(\underline{\mathrm{Hom}}_*(S^n, R)).\end{aligned}$$

Then we define the associated graded ring as

$$\pi_*(R) := \bigoplus_{n \geq 0} \pi_n(R)$$

which is a graded ring because there are maps

$$\underline{\mathrm{Hom}}_*(S^n, R) \times \underline{\mathrm{Hom}}_*(S^m, R) \to \underline{\mathrm{Hom}}_*(S^n \times S^m, R \times R) \to \underline{\mathrm{Hom}}_*(S^n \wedge S^m, R)$$

where the last map is induced by multiplication. If we now take connected components then we get maps

$$\pi_n(R) \times \pi_m(R) \to \pi_{n+m}(R).$$

Definition 2. *Let k be a field considered as a discrete simplicial set, then we define the category $\mathbf{sArt_k}$ of simplicial Artin local rings as the full subcategory of $\mathbf{sCR}/k$ (simplicial commutative rings with a fixed map to k) on the objects R satisfying:*

- *The discrete ring $\pi_0(R)$ is an Artian local ring with residue field k*
- *The associated graded ring $\pi_*(R)$ is a finitely generated $\pi_0(R)$ module.*

4 Deformation problems

We will study functors $\mathcal{F} : \mathbf{sArt_k} \to \mathbf{sSets}$.

Definition 3. *We call $\mathcal{F}$ **homotopy invariant** if it preserves weak equivalences. A **simplicial enrichment** of $\mathcal{F}$ is a choice of morphisms*

$$\underline{\mathbf{sArt_k}}(R, S) \to \underline{\mathrm{Hom}}(\mathcal{F}(R), \mathcal{F}(S))$$

for each $R, S \in \mathbf{sArt_k}$ which is compatible with compositions and extending the usual functoriality of $\mathcal{F}$ on zero simplices.

Lemma 1. *Important example: If $R \in \mathbf{sArt_k}$ is cofibrant, then*

$$\underline{\mathbf{sArt_k}}(R, -)$$

is simplicially enriched and homotopy invariant.

Proof. Simplicial enrichment: For $S, T \in \mathbf{sArt_k}$ we want to define a map

$$\underline{\mathbf{sArt_k}}(S, T) \to \underline{\mathrm{Hom}}(\underline{\mathbf{sArt_k}}(R, S), \underline{\mathbf{sArt_k}}(R, T)).$$

By adjunction this would correspond to a map (by the exponential law)

$$\underline{\mathbf{sArt_k}}(S, T) \times \underline{\mathbf{sArt_k}}(R, S) \to \underline{\mathbf{sArt_k}}(R, T)$$

which we can take to be the composition morphism, which clearly extends the usual functoriality on zero simplices. For homotopy invariance we note the following: Since every simplicial commutative ring is fibrant, every weak equivalence is a weak equivalence between fibrant objects. By Ken Brown's Lemma, it suffices to show the functor $\underline{\mathbf{sArt_k}}(R,-)$ preserves trivial fibrations. So let $f : S \to T$ be such a trivial fibration and $X \to Y$ a cofibration between simplicial sets. Then we want to show that any diagram

$$\begin{array}{ccc} X & \longrightarrow & \underline{\mathbf{sArt_k}}(R,S) \\ \big\downarrow & \nearrow & \big\downarrow f \\ Y & \longrightarrow & \underline{\mathbf{sArt_k}}(R,T) \end{array}$$

has a lifting, proving that f is a trivial fibration. We claim that the lifting in the diagram is equivalent to a lift in the following diagram (using the exponential law)

$$\begin{array}{ccc} & & S^Y \\ & \nearrow & \big\downarrow \\ R & \longrightarrow & S^X \times_{T^X} T^Y. \end{array}$$

But since $X \to Y$ is cofibrant and $S \to T$ is a trivial fibration we find that the vertical map is a trivial fibration (by the important fact stated in the beginning). We conclude that a lift exists since R is cofibrant.

Definition 4. *A natural weak equivalence $\eta : \mathcal{F} \to \mathcal{G}$ between functors $\mathcal{F}, \mathcal{G} : \mathbf{sArt_k} \to \mathbf{sSets}$ is a natural transformation such that all components*

$$\eta_R : \mathcal{F}(R) \to \mathcal{G}(R)$$

are weak equivalences. The functors $\mathcal{F}, \mathcal{G}$ are called naturally weakly equivalent if there is a zig-zag of natural weak equivalences.

Lemma 2 (Technical Lemma)**.** *If $\mathcal{F}$ is homotopy invariant, then there exists an $\mathcal{F}'$, which is simplicially enriched and has values in Kan complexes, and a natural weak equivalence*

$$\mathcal{F} \to \mathcal{F}'.$$

Moreover we can make $\mathcal{F} \to \mathcal{F}'$ functorial in $\mathcal{F}$.

The importance of this functoriality is that we can replace a zig-zag of homotopy invariant functors by a a weakly equivalent zig-zag such that all functors (except possibly the endpoints) are simplicially enriched and Kan valued. □

Definition 5. *We call a functor $\mathcal{F} : \mathbf{sArt_k} \to \mathbf{sSets}$* ***representable*** *if it is naturally weakly equivalent to $\underline{\mathbf{sArt_k}}(R,-)$ for some cofibrant $R \in \mathbf{sArt_k}$.*

We remark that any representable functor is homotopy invariant, since $\underline{\mathbf{sArt_k}}(R,-)$ is and homotopy invariance is preserved by natural weak equivalence.

If $\mathcal{F}, \mathcal{G}$ are simplicially enriched, then there is a simplicial set $\underline{\mathrm{Nat}}(\mathcal{F}, \mathcal{G})$ whose simplices are described by

$$\underline{\mathrm{Nat}}(\mathcal{F}, \mathcal{G})_n := \{\text{Natural transformations } \Delta^n \times \mathcal{F} \to \mathcal{G}\}$$

where Δ^n denotes the constant functor with value Δ^n. We get an enriched Yoneda lemma:

$$\underline{\mathrm{Nat}}(\underline{\mathbf{sArt_k}}(R,-),\mathcal{F}) \cong \mathcal{F}(R).$$

Proposition 1. *If $\mathcal{F}$ is simplicially enriched then $\mathcal{F}$ is representable if and only if there exists a cofibrant R and a vertex $v \in \mathcal{F}(R)_0$ such that the corresponding map (coming from the enriched Yoneda Lemma)*

$$\underline{\mathbf{sArt_k}}(R,-) \to \mathcal{F}$$

is a natural weak equivalence.

Proof. If there is such a vertex, then $\mathcal{F}$ is representable by definition.

Now first suppose that there is a natural weak equivalence $\eta : F \to \underline{\mathbf{sCR}}(R,-)$. Choose $v \in \mathcal{F}(R)_0$ such that $\eta(v)$ is in the same connected component as the identity in $\underline{\mathbf{sCR}}(R,R)_0$, let

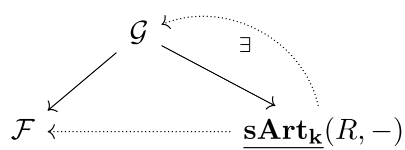
$$\nu : \underline{\mathbf{sArt_k}}(R,-) \to \mathcal{F}$$

be the corresponding map (under enriched Yoneda).

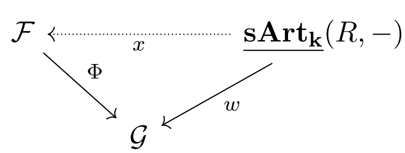
Then $\eta\circ\nu : \underline{\mathbf{sArt_k}}(R,-) \to \underline{\mathbf{sArt_k}}(R,-)$ corresponds to $\eta(v)$. Since $\underline{\mathbf{sArt_k}}(R,R)$ is Kan there is an actual homotopy H between $\eta(v)$ and the identity map. By simplical Yoneda this correspond to a homotopy

$$\Delta^1 \times \underline{\mathbf{sArt_k}}(R,-) \to \underline{\mathbf{sArt_k}}(R,-)$$

between the identity natural transformation and $\eta \circ \nu$. This implies that ν is a natural weak equivalence. This basically means that we can now get rid of hats .

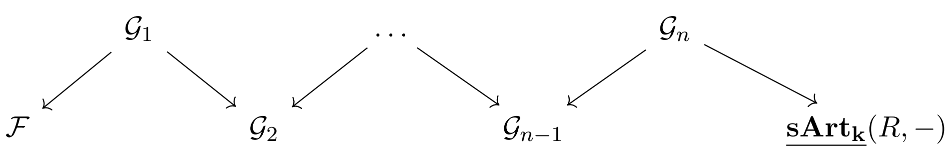


Now suppose we have



with $\mathcal{G}$ Kan valued and simplicially enriched. Then we choose $x \in \mathcal{F}(R)_0$ such that $\Phi(x) \sim w$. Since $\mathcal{G}(R)$ is Kan we can find a homotopy $H : \Delta^1 \to \mathcal{G}(R)$ between $\phi(x)$ and w which shows that x is a natural weak equivalence.

For the general case, we have a zig-zag of natural weak equivalences



All the functors in this zig-zag are homotopy invariant (because representable) and hence by the Technical Lemma we may assume that $\mathcal{G}_1, ...\mathcal{G}_n$ are simplicially enriched and Kan valued. We may then argue by induction on n using the two cases above. □

Definition 6. *We say that a functor $\mathcal{F} : \mathbf{sArt_k} \to \mathbf{sSets}$ is **pro-representable** if there is a filtered category J and a pro-object (with R_j cofibrant)*

$$D : J \to \mathbf{sArt_k}$$
$$j \mapsto R_j$$

such that $\mathcal{F}$ is naturally weakly equivalent to the functor

$$\mathrm{colim}_{J^{op}}\, \underline{\mathbf{sArt_k}}(R_j, -).$$

*The functor $\mathcal{F}$ is **sequentially pro-representable** if we can choose $J = (\mathbb{N}, <)$ by which we mean the category*

$$\{\cdots \to * \to *\}.$$