# DERIVED GALOIS DEFORMATION RINGS

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## 1. Simplicial Galois Representations

1.1. **Motivation.** First we need to discuss how to define a derived version of a deformation of a Galois representation. Since the coefficients are now allowed to be simplicial Artin rings, we need a new definition. So what do we mean by

$$G_{K,S} \to G(A)?$$

Here $G_{K,S}$ is the Galois group of the maximal algebraic extension of $K$ unramified outside a finite set of place $S$, $G$ is a reductive group (later we may take $G$ to be adjoint), and $A \in \mathrm{sArt}_k$ (the category of simplicial local Artin rings, as defined in Raffael's talk).

The naive idea (which doesn't work) is to define $G(A)$ directly.

(1) Let $A \in \mathrm{sArt}_k$. We could just say $[p] \mapsto G(A_p)$, which will define a simplicial group. Unfortunately, this is not homotopy invariant: to see this note that

$$G(A_p) = \mathrm{Hom}_{\mathrm{CR}}(\mathscr{O}_G, A_p) = \mathrm{Hom}_{\mathrm{sCR}}(\mathscr{O}_G, A^{\Delta^p}) = \underline{\mathrm{Hom}}_{\mathrm{sCR}}(\mathscr{O}_G, A)_p,$$

where we view $\mathscr{O}_G$ as a constant simplicial ring in the third and fourth term and the underline denotes simplicial enrichment. Therefore, our attempt is just

$$A \mapsto \underline{\mathrm{Hom}}_{\mathrm{sCR}}(\mathscr{O}_G, A).$$

But $\mathscr{O}_G$ (viewed as a discrete simplicial ring) will almost never be cofibrant, so there's no reason to expect that we should get something homotopy invariant.

(2) So instead we could define $G(A) := \underline{\mathrm{Hom}}_{\mathrm{sCR}}(c(\mathscr{O}_G), A)$. This is now homotopy invariant, but unfortunately it's not a simplicial group, because cofibrant replacement won't respect the Hopf algebra structure of $\mathscr{O}_G$, so this isn't quite what we want.

(3) (comment/speculation from the audience) maybe we could put a model structure on the category of simplicial Hopf algebras and then try to cofibrantly replace $\mathscr{O}_G$, now viewed as a constant simplicial Hopf algebra? Unclear.

So instead of trying to define $G(A)$ directly, we make the observation that actually $G_{K,S} = \pi_1^{\acute{e}t}(\mathbf{Z}[\frac{1}{S}], *)$. But this profinite group can be viewed as the fundamental group of a pro-(pointed simplicial set) $X$ (in fact there are two ways of doing this, which we will describe in a moment). Then we make the observation that in the discrete case, i.e when $A$ is an ordinary ring,

$$\{\rho : G_{K,S} \to G(A)\} = \{G(A)\text{-torsors over } |X|\} = \mathrm{Hom}_{\mathrm{Top}}(|X|, BG(A))/\sim$$

where $|X|$ denotes the geometric realization of the pro-(simplicial set) $X$, $BG(A)$ is the classifying space of the group $G(A)$, and $\sim$ means that we're taking homotopy classes of morphisms.

So what are these spaces $X$? One candidate is the étale topological type defined in [**?** ] following [**?** ]. This is a pro-(simplicial set) $(X_i)_i$ indexed by étale hypercoverings of the scheme $\mathrm{Spec}\,\mathbf{Z}[\frac{1}{S}]$, whose $\pi_1(X, *) :=$

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$\lim_i \pi_1(X_i, *)$ recovers the étale fundamental group of $\operatorname{Spec} \mathbf{Z}[\frac{1}{S}]$. For our purposes we can do something simpler, which is to note that $G_{K,S} = \lim_\alpha G_\alpha$ is a profinite group, and then we can take $X$ to be the pro-system $(X_\alpha)_\alpha = (N(G_\alpha))_\alpha$, where $N$ denotes the nerve of a group, viewed as a one-object groupoid.

### 1.2. Defining $BG$.

So we now need to define some notion of $BG(A)$ for $A \in \mathrm{sArt}_k$. For ordinary commutative rings $A$, note $BG(A)$ is the geometric realization of the nerve of $G(A)$: i.e. if $N_p(G(A))$ denotes the $p$-simplices of the nerve, then the functor of points

$$A \mapsto N_p G(A)$$

is represented by $G^{\times p}$. Why is this true? To construct $BG$ for a discrete group, we construct $EG$ a contractible space and has a free action of $G$, and then we take $BG = EG/G$.

To do this, let $C$ be the category whose objects are indexed by elements of $G$, and whose morphisms are $g \to gh$. Let $D$ have one object, with morphisms labelled by $G$ and composition is multiplication. Then there's a map $C \to D$.

In general, if $C$ is a small category, then $NC$ is a simplicial set where the 0-simplices are objects of $C$, and for $k > 0$, the $k$-simplices are $k$-tuples of composable morphisms.

So essentially, the nerve of $C$ (above) is contractible, and if we quotient by $G$, then we get the nerve of $D$.

With this in mind, we can now define $BG$ for a simplicial ring.

**Definition 1.1.** Consider the bisimplicial set $[p] \mapsto \underline{\mathrm{Hom}}_{\mathrm{sCR}}(c(\mathscr{O}_{N_pG}), A)$. Then $BG(A)$ is $\mathrm{Ex}^\infty$ (fibrant replacement) of the geometric realization of $\underline{\mathrm{Hom}}_{\mathrm{sCR}}(c(\mathscr{O}_{N_pG}), A)$: note the geometric realization can be computed either by taking the "total simplicial set" of the bisimplicial set, or by taking the diagonal: in fact these are homotopy equivalent (this is not easy: see [**?** ]).

Concretely, if $A$ is discrete, then $BG(A)$ is weakly equivalent to $NG(\pi_0(A))$. In the definition, we need the cofibrant replacements and the $\mathrm{Ex}^\infty$ fibrant replacement in order for this thing to behave well, at least homotopy theoretically.

### 1.3. Galois Deformations.

Now we can talk about Galois deformations. So let $(X_\alpha)_\alpha$ be either the étale topological type for $\operatorname{Spec} \mathbf{Z}[\frac{1}{S}]$, or the pro-simplicial set $NG_\alpha$ where $\alpha$ varies over the finite Galois groups.

**Definition 1.2.** Now fix a map $\overline{\rho} : X \to BG(k)$ in $\mathrm{pro-sSet}$. Then define the *unframed deformation functor*

$$F_{\mathbf{Z}[\frac{1}{S}],\overline{\rho}} = \underline{\mathrm{Hom}}_{\mathrm{pro-sSet}}(X, BG(A)) \times^h_{\underline{\mathrm{Hom}}_{\mathrm{pro-sSet}}(X, BG(k))} \overline{\rho}$$

where $\underline{\mathrm{Hom}}_{\mathrm{pro-sSet}}(X, BG(A)) = \mathrm{colim}_\alpha \underline{\mathrm{Hom}}_{\mathrm{sSet}}(X_\alpha, BG(A))$ and $\overline{\rho}$ is really $\Delta^0$ with the map to $\underline{\mathrm{Hom}}_{\mathrm{pro-sSet}}(X, BG(k))$ given by $\overline{\rho}$. There is also a framed version, where one replaces $\mathrm{pro-sSet}$ with $\mathrm{pro-sSet}_*$, the pro-category of pointed simplicial sets (and choosing basepoints for $X$ and $BG$). Keeping track of this basepoint can be roughly thought of as keeping track of a basis, which explains why this is the framed thing.

## 2. Pro-Representability

Recall the derived Schlessinger criterion from last week. This says that if $F : \mathrm{sArt}_k \to \mathrm{sSet}$ is formally cohesive, then it is pro-representable if and only if $\pi_i(\mathfrak{t}F) = 0$ for $i > 0$, where $\mathfrak{t}F$ is the tangent complex of $F$ as defined by Dougal last week.

But in our situation, $BG(k)$ will not be contractible, so $BG$ won't be formally cohesive. So instead of having a tangent complex, we get a local system $\mathfrak{t}BG$ on $BG(k)$ , i.e. a functor $L : \mathrm{Simp}(BG(k)) \to \mathrm{Ch}(k)$ sending all morphisms to quasi-isomorphisms (this was defined by Dougal last week). Recall the following result:

**Proposition 2.1.** *If $F : \mathrm{sArt}_k \to \mathrm{sSet}$ is now any homotopy invariant functor which preserves pullbacks (in the sense of Dougal's talk), and given $\overline{\rho} : X \to F(k)$, consider the new functor*

$$F_{X,\overline{\rho}}(A) := \mathrm{hofib}_{\overline{\rho}}(\underline{\mathrm{Hom}}_{\mathrm{sSet}}(X, F(A)) \to \underline{\mathrm{Hom}}_{\mathrm{sSet}}(X, F(k))).$$

*This is formally cohesive, and the tangent complex is*

$$\mathfrak{t}F_{X,\overline{\rho}} \cong C^*(X, \overline{\rho}^*\mathfrak{t}F)$$

*where $C^*$ is the cochains construction introduced last week.*

So all we need to know is that $BG$ is homotopy invariant and preserves homotopy pullbacks, and then we can hope to apply the Derived Schlessinger Criterion by computing the homotopy groups of $\mathfrak{t}F_{X,\overline{\rho}}$.

Note $BG$ is homotopy invariant because of the fibrant replacement we took in the definition. There's a criterion to check that $BG$ preserves homotopy pullbacks.

**Proposition 2.2.** *If $F : \mathrm{sArt}_k \to \mathrm{sSet}$ is homotopy invariant, $F(A)$ is path-connected for all $A$, $A \mapsto \Omega F(A)$ (loop space) preserves homotopy pullbacks, and $\pi_0\Omega F(A) \to \pi_0\Omega F(B)$ is surjective whenever $\pi_0 A \to \pi_0 B$ is surjective, then $F$ preserves homotopy pullbacks.*

To apply this, we use that $G(A) := \underline{\mathrm{Hom}}_{\mathrm{sCR}}(c(\mathscr{O}_G), A) \to \Omega BG(A)$ is a weak equivalence, which should heuristically be true by looking at the homotopy groups.

**Lemma 2.3.** *The tangent complex of $A \mapsto BG(A)$ is a local system on $BG(k)$ whose homology is $\mathfrak{g}$, the Lie algebra of $G(k)$, concentrated in degree 1 with a $G(k)$-action (via the adjoint action, conjugation) at a basepoint.*

Note the $G(k)$-action arises because for any $Z$ a simplicial set and $L : \mathrm{Simp}(Z) \to \mathrm{Ch}(k)$ a local system, one can check directly that $\pi_1(Z, z)$ naturally acts on $H_*(L_z)$, where $z : \Delta^0 \to Z$ is some basepoint.

**Proposition 2.4.** *The tangent complex $\mathfrak{t}F_{\mathbf{Z}[\frac{1}{S}],\overline{\rho}}$ is quasi-isomorphic to $C^{*+1}(X, \overline{\rho}^*\mathfrak{g})$, and*

$$\pi_{-i}(\mathfrak{t}F_{\mathbf{Z}[\frac{1}{S}],\overline{\rho}}) \cong H^{i+1}(X, \overline{\rho}^*\mathfrak{g}) = H^{i+1}(\mathbf{Z}[\frac{1}{S}], \mathrm{ad}\,\overline{\rho})$$

*for $i \geq -1$ (for $i > 1$ we have $\pi_i(\mathfrak{t}F_{\mathbf{Z}[\frac{1}{S}],\overline{\rho}}) = 0$).*

The last identification with étale cohomology (i.e. continuous group cohomology in this case) can be seen by identifying the cochains construction with étale cochains.

So if $G$ is an adjoint group (i.e. has trivial centralizer) and $\overline{\rho}$ is Schur (i.e. the centralizer of $\overline{\rho}$ is the center of the group), then this is telling us that $H^0(\mathbf{Z}[\frac{1}{S}], \mathrm{ad}\,\overline{\rho}) = 0$, so we're pro-representable by derived Schlessinger's criterion. In general, one can modify this construction to take into account groups whose center is non-trivial (like $\mathrm{GL}_n$): for the purposes of this study group, we'll ignore this, but the details are worked out in Section 5.4 of [**?** ].

**Lemma 2.5.** *The functor $\pi_0 F_{Z[\frac{1}{S}],\overline{\rho}} : \mathrm{Art}_k \to \mathsf{Set}$ is isomorphic to the usual deformation functor if $\overline{\rho}$ is Schur, i.e. the centralizer of $\overline{\rho}$ is $Z(G)$.*

We get a similar result for the framed deformations, without assuming the Schur condition.

*Proof.* This is basically unwinding definitions. We're asking about components of $\mathrm{Hom}_{\mathrm{sSet}}(X, BG(A))$, which correspond to isomorphism classes of $G(A)$-torsors over $X$, which in turn correspond to conjugacy classes of Galois representations $\overline{\rho} : Z[1/S] \to G(A)$.

To dig a bit into why this should be true, consider the following equalities in the framed case. Suppose $A \in \mathrm{Art}_k$ is an ordinary (underived) Artin ring with residue field $k$. Then if $G_{K,S} = \lim_\alpha G_\alpha$

$$\begin{aligned}\pi_0 \underline{\mathrm{Hom}}_{\mathrm{pro-(sSet_*)}}((N(G_\alpha),*)_\alpha,(BG(A),*)) &= \pi_0 \operatorname{colim}_\alpha \underline{\mathrm{Hom}}_{\mathrm{sSet}_*}((N(G_\alpha),*),(BG(A),*)) \\ &= \operatorname{colim}_\alpha \pi_0 \underline{\mathrm{Hom}}_{\mathrm{sSet}_*}((N(G_\alpha),*),(BG(A),*)) \\ &= \operatorname{colim}_\alpha \pi_0 \underline{\mathrm{Hom}}_{\mathrm{sSet}_*}((N(G_\alpha),*),(N(G(A)),*)) \\ &= \operatorname{colim}_\alpha \mathrm{Hom}_{\mathrm{Grp}}(G_\alpha, G(A)) \\ &= \mathrm{Hom}_{\mathrm{pro-(FinGrp)}}((G_\alpha)_\alpha, G(A)) \\ &= \mathrm{Hom}_{\mathrm{cont}}(\varprojlim_\alpha G_\alpha, G(A)).\end{aligned}$$

The first equality is the definition of Hom sets in the pro-category, the second is the fact that $\pi_0$ commutes with filtered colimits, the third is the equivalence of $BG(A)$ with $N(G(A))$ when $A$ is discrete, the fourth is the adjunction between $\pi_1$ and the classifying space (in the homotopy category), the fifth is the definition of Hom in a pro-category again, and the sixth is the fact that pro-(finite groups) are the same as profinite groups with the profinite topology. □

## 3. Local Conditions

Let $\overline{\rho} : \pi_1 \operatorname{Spec} \mathbf{Z}[\frac{1}{S}] \to G(k)$ be a fixed Galois representation. If $v \in S$ is some finite place, let $F_{\mathbf{Q}_v,\overline{\rho}}$ denote the deformation functor for $\overline{\rho}$ pulled back to $\pi_1 \operatorname{Spec} \mathbf{Q}_v$. We then get a natural transformation

$$F_{\mathbf{Z}[\frac{1}{S}],\overline{\rho}} \to F_{\mathbf{Q}_v,\overline{\rho}}$$

**Definition 3.1.** A local condition is a simplicially enriched functor $D_v : \mathrm{sArt}_k \to \mathrm{sSet}$ equipped with a natural transformation

$$D_v \to F_{\mathbf{Q}_v,\overline{\rho}}.$$

The corresponding global deformation functor with local conditions is defined to be

$$F^D_{\mathbf{Z}[\frac{1}{S}],\overline{\rho}} := F_{\mathbf{Z}[\frac{1}{S}],\overline{\rho}} \times^h_{F_{\mathbf{Q}_v,\overline{\rho}}} D_v$$

**Remark 3.2.** We don't necessarily need a map $D_v \to F_{\mathbf{Q}_v,\overline{\rho}}$: we can take a zig-zag instead, where the maps going the "wrong way" are weak equivalences, and still make the theory work. See the remark after (9.1) in [**?** ].

**Example 3.3** (Sanity Check)**.** Suppose $\overline{\rho}$ is actually unramified at $v$, and let $S' = S \setminus \{v\}$. Then we have a natural transformation

$$F_{\mathbf{Z}_v,\overline{\rho}} \to F_{\mathbf{Q}_v,\overline{\rho}}.$$

If we take $D_v = F_{\mathbf{Z}_v,\overline{\rho}}$, then the global deformation functor

$$F^D_{\mathbf{Z}[\frac{1}{S}],\overline{\rho}} = F_{\mathbf{Z}[\frac{1}{S}],\overline{\rho}} \times^h_{F_{\mathbf{Q}_v,\overline{\rho}}} D_v$$

is weakly equivalent to $F_{\mathbf{Z}[\frac{1}{S'}],\overline{\rho}}$: in [**?** ] they prove this by noting that each functor is formally cohesive, so it suffices to check that the induced fiber sequence of tangent complexes is an isomorphism: see Section 8 of their paper for the details.

In practice, Galatius and Venkatesh want to turn underived local conditions into derived local conditions. Assume $F_{\mathbf{Q}_v,\overline{\rho}}$ is pro-representable (this is the only case they will care about later) with simplicial pro-ring $\mathcal{R}_v$. Then we have maps

$$\mathcal{R}_v \to \pi_0 \mathcal{R}_v =: R_v \twoheadrightarrow R_v^D,$$

where $\overline{R_v^D}$ is the underived local condition. Now let

$$D_v := \mathrm{Hom}(c(R_v^D), -).$$

We then get a zig-zag

$$\mathcal{R}_v \xleftarrow{\sim} c(\mathcal{R}_v) \to c(R_v) \to c(R_v^D),$$

and by taking Hom we get

$$\underline{\mathrm{Hom}}_{\mathrm{sSet}}(c(R_v^D), -) \to \underline{\mathrm{Hom}}_{\mathrm{sSet}}(c(R_v), -) \to \underline{\mathrm{Hom}}_{\mathrm{pro-sSet}}(c(\mathcal{R}_v)), -) \xleftarrow{\sim} \underline{\mathrm{Hom}}_{\mathrm{pro-sSet}}(\mathcal{R}_v, -).$$

Now use Remark 3.2 to obtain a local condition.

**Theorem 3.4.** *Suppose $R_v^D$ is formally smooth. Then $\mathfrak{t}^i F^D_{\mathbf{Z}[\frac{1}{S}],\overline{\rho}} \cong H_D^{i+1}(\mathbf{Z}[\frac{1}{S}], \mathrm{ad}\,\overline{\rho})$.*

*Proof Sketch.* We have a map $\mathfrak{t}D_v \to \mathfrak{t}F_{\mathbf{Q}_v,\overline{\rho}}$ and a quasi-isomorphism $\tau_{\geq 0}(\mathfrak{t}D_v) \to \mathfrak{t}D_v$. Therefore,

$$\mathfrak{t}F^D_{\mathbf{Z}[\frac{1}{S}],\overline{\rho}} \xrightarrow{\sim} \mathrm{hofib}(\mathfrak{t}F_{\mathbf{Z}[\frac{1}{S}],\overline{\rho}} \oplus \tau_{\geq 0}(\mathfrak{t}D_v) \to \mathfrak{t}F_{\mathbf{Q}_v,\overline{\rho}}).$$

is a natural weak equivalence.

But we have a factorization $\tau_{\geq 0}(D_v) \to \tau_{\geq 0}(\mathfrak{t}F_{\mathbf{Q}_v,\overline{\rho}}) \to \mathfrak{t}F_{\mathbf{Q}_v,\overline{\rho}}$. The source and target of the first map have homotopy only in degree 0, so the first map induces a quasi-isomorphism onto the subcomplex $\tau_{\geq 0}(\mathfrak{t}F_{\mathbf{Q}_v,\overline{\rho}})$ whose cohomology is $H^1_D(\mathbf{Q}_v, \mathrm{ad}\,\overline{\rho})$. Note the fact that this is true is not obvious □

Going forward, we have some extra assumptions on $\overline{\rho}$:

(1) $H^0(\mathbf{Q}_p, \mathrm{ad}\,\overline{\rho}) = H^2(\mathbf{Q}_p, \mathrm{ad}\,\overline{\rho}) = 0$: this means that at $p$, the universal deformation problem is pro-representable and formally smooth.

(2) For $v \in S \setminus \{p\}$, $H^j(\mathbf{Q}_v, \mathrm{ad}\,\overline{\rho}) = 0$ for $j = 0, 1, 2$: this means that we have trivial deformation theory away from $p$ in $S$.

(3) (big image) The image of $\overline{\rho}|_{Q(\zeta_{p^\infty})}$ contains the image of $G^{sc}(k)$ (simply connected cover) in $G(k)$.

(4) At $p$, $\overline{\rho}$ is torsion crystalline, and there is an unobstructed subfunctor $\mathrm{Def}^{\mathrm{cris}} \subset \mathrm{Def}_p$ such that the tangent space is $H^1_f(\mathbf{Q}_v, \mathrm{ad}\,\overline{\rho})$.