COHOMOLOGY OF ARITHMETIC GROUPS

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Let G be a semisimple real lie group with Γ a discrete cocompact [1] subgroup that is torsion free. Let K be a maximal compact subgroup of G and let V be a finite dimensional complex continuous representation of G that we might assume irreducible. We are interested in studying the cohomology groups

$$H^*(\Gamma, V),$$

which, as we have seen in the previous lectures, are closely related to the theory of automorphic forms. We will show the following results:

- The cohomology groups can be computed as
$$H^n(\Gamma, V) = H^k(\mathfrak{g}, K; \mathcal{C}^\infty(\Gamma\backslash G) \otimes V).$$
- These $(\mathfrak{g}, K)$-cohomology groups can be decomposed as
$$H^k(\mathfrak{g}, K; \mathcal{C}^\infty(\Gamma\backslash G) \otimes V = \bigoplus_\pi m(\pi, \Gamma)\ H^n(\mathfrak{g}, K; H_\pi \otimes V),$$
where (π, H_π) runs over certain representations of G. This is known as Matsushima's formula.
- We will finally study the groups
$$H^k(\mathfrak{g}, K; H_\pi \otimes V)$$
in more detail. In particular, we will show that, for certain representations π, they vanish outside certain range and we will calculate their dimension.

1. Cohomology and differential forms

Reference: [BW00, §VII.2]

Let $X := G/K$ denote the symmetric space associated to G, it is simply connected and contractible.

1.1. **Differential forms.** Let $\mathcal{A}^q = \mathcal{A}^q(X, V)$ denote the smooth V-valued differentials forms of degree q on X with the usual differentials $d : \mathcal{A}^q \to \mathcal{A}^{q+1}$ given by

$$d\omega(v_1, \dots, v_q) = \sum_{i=1}^{q} (-1)^i v_i \cdot \omega(v_1, \dots, \hat{v}_i, \dots, v_q) + \sum_{i<j} (-1)^{i+j} \omega([v_i, v_j], v_1, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots v_q),$$

where $v_i \cdot \omega(v_1, \dots, \hat{v}_i, \dots, v_q)$ denotes the differentiation of the function ω in the direction v_i, $[,]$ refers to the bracket of vector fields, and $\hat{}$ means omission of the corresponding argument.

[1] We do not say anything about the non-compact case due to lack of time, but that case is of particular interest. See [BW00, §XIV] for the analogous statements in this context.

Proposition 1. *There is an canonical isomorphism*

$$H^*(\Gamma, V) = H^*(\Gamma\backslash X, \tilde{V})$$

where $\tilde{V}$ is the local system on $\Gamma\backslash X$ associated to V.

Proof. This follows immediately from the fact that $\Gamma\backslash X$ is a $K(\Gamma,1)$-space, i.e. $\pi_1(\Gamma\backslash X) = \Gamma$ and all its other homotopy groups vanish. □

The comparison between de Rham and singular cohomology now gives us the following Corollary:

Corollary 1. *There are canonical isomorphisms*

$$H^*(\Gamma, V) \cong H^*\big(\mathcal{A}^\bullet(\Gamma\backslash X, \tilde{V})\big) \cong H^*\big(\mathcal{A}^\bullet(X, \tilde{V})^\Gamma\big).$$

1.2. $(\mathfrak{g}, K)$ **modules and** $(\mathfrak{g}, K)$**-cohomology.** Let $\mathfrak{g}$ denote the Lie algebra of G. Recall that a $(\mathfrak{g}, K)$-module is a vector space W over $\mathbb{R}$ which is a $\mathfrak{g}$-module and a K-module with the obvious compatibility condition. Namely we ask

- $\pi(k)\cdot(\pi(X)\cdot v) = \pi(\mathrm{Ad}\, k(X))\cdot(\pi(k)\cdot v)$, for all $k \in K, X \in \mathfrak{g}, v \in W$,
- If $F \subseteq W$ is a K-stable finite dimensional subspace, then the representation of K is differentiable, and has $\pi|_{\mathfrak{k}}$ as differential.

Example 1. If V is a representation of G, then the subspace of smooth and K-finite vectors of V is a $(\mathfrak{g}, K)$-module.

Definition 1. *For V a $(\mathfrak{g}, K)$-module we define*

$$\mathcal{C}^q(\mathfrak{g}, K; V) = \mathrm{Hom}_K(\wedge^q(\mathfrak{g}/\mathfrak{k}, V) = (\wedge^q \mathfrak{p}^* \otimes V)^{K/K^0}$$

where $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition. There are differentials defined in the same way as was done in §1.1 which gives us a complex $\mathcal{C}^\bullet(\mathfrak{g}, K; V)$ and we define the $(\mathfrak{g}, K)$-cohomology of V as the cohomology groups of this complex:

$$H^*(\mathfrak{g}, K; V) := H^*(\mathcal{C}^\bullet(\mathfrak{g}, K; V)).$$

The left translation by elements $g \in G$ provides an isomorphism between the tangent space at g and the tangent space at the identity element, and hence an identification

$$\mathcal{A}^q(\Gamma\backslash X, V) = \mathrm{Hom}_K(\wedge^q(\mathfrak{g}/K, \mathcal{C}^\infty(\Gamma\backslash G)V) = \mathcal{C}^q(\mathfrak{g}, K; \mathcal{C}^\infty(\Gamma\backslash G) \otimes V).$$

An explicit computation of the differentials now gives:

Proposition 2. *There is a canonical isomorphism*

$$H^*(\Gamma\backslash X, \tilde{V}) = H^*(\mathfrak{g}, K; \mathcal{C}^\infty(\Gamma\backslash G) \otimes V)$$

2. Matsushima's Formula

Reference: [BW00, §VII.3-6]

Let $L^2(\Gamma\backslash G, V)$ be the space of square-integrable V-valued functions of $\Gamma\backslash G$. This is acted upon by G and decomposes as

$$L^2(\Gamma\backslash G) = \widehat{\bigoplus_{\pi}} m(\pi,\Gamma)\cdot H_\pi,$$

a direct sum of irreducible representations with finite multiplicities. Moreover one has

$$\mathcal{C}^\infty(\Gamma\backslash G) = \left(L^2(\Gamma\backslash G)\right)^\infty = \left(\widehat{\bigoplus_{\pi}} m(\pi,\Gamma)\cdot H_\pi\right)^\infty.$$

where $(-)^\infty$ means taking smooth vectors.

Proposition 3 (Matsushima's formula, [BW00, VII.3.2 Theorem])**.** *We have*

$$H^*(\mathfrak{g}, K; \mathcal{C}^\infty(\Gamma\backslash G)\otimes V) = \bigoplus_{\pi} m(\pi,\Gamma)\cdot H^*(\mathfrak{g},K;H_\pi\otimes V)$$

where the direct sum is now finite.

Proof. The previously stated facts give us

$$H^*(\mathfrak{g}, K; \mathcal{C}^\infty(\Gamma\backslash G)\otimes V) = H^*(\mathfrak{g},K;(\widehat{\bigoplus_{\pi}} m(\pi,\Gamma)\cdot H_\pi)^\infty\otimes V).$$

We want to show that the right hand side term equals

$$\bigoplus_{\pi} m(\pi,\Gamma)H^*(\mathfrak{g},K;H_\pi\otimes V).$$

Let now $S\subseteq \hat{G}$ be a finite set of representations. Then we can decompose

$$H^*(\Gamma, V) = \bigoplus_{\pi\in S} m(\pi,\Gamma)H^*(\mathfrak{g},K;H_\pi\otimes V)\oplus H^*(\mathfrak{g},K;(\widehat{\bigoplus_{\pi\notin S}} m(\pi,\Gamma)\cdot H_\pi)^\infty\otimes V).$$

The compactness assumption on Γ tells us that the cohomology $H^*(\Gamma, V)$ of the arithmetic group is finite dimensional. We deduce, for dimension reasons, that, for a large enough S, we have

$$H^*(\mathfrak{g},K;H_\pi\otimes V) = 0 \quad \forall \pi\notin S.$$

We have hence reduced then to proving that, if each $(\mathfrak{g}, K)$-cohomology of a countable collection of irreducible unitary representations of G vanishes, then the cohomology of its closed direct sum vanishes as well. This is not very hard and follows from a topological argument (cf. [BW00, VII.3.3 Lemma] for the details). ☐

Summarising , we have reduced our computation of $H^*(\Gamma, V)$ to the study the $(\mathfrak{g}, K)$ cohomology groups of certain representations of the form $H_\pi\otimes V$, where H_π is a unitary $(\mathfrak{g}, K)$-module and V is a finite dimensional (irreducible) complex continuous representation of G..

Remark 1. We will give later a necessary condition for π to appear in the the above sum. A precise characterisation of the representations π that contribute to $H^*(\Gamma, V)$ have been described by Vogan-Zuckerman.

3. Calculation of the $(\mathfrak{g}, K)$-cohomology

Reference: [BW00, §II].

Let (ρ, E) be a finite dimensional irreducible complex representations of G [2] and (σ, H) be a unitary $(\mathfrak{g}, K)$-module. Let $V = H \otimes E$ and $\tau = \rho \otimes \sigma$. With an eye on Matsushima's formula, we want to study the cohomology groups $H^*(\mathfrak{g}, K; V)$ in this particular case.

3.1. The Casimir element.

Let (y_i) be a basis of $\mathfrak{g}$ and (y_i') be its dual basis with respect to the Killing form. Then

$$C = \sum_i y_i \cdot y_i'$$

is an element of the center of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$, independent of the choice of basis, and is called the Casimir element. By Schur's lemma, C must act as a scalar on any representation.

Proposition 4 ([BW00, §II.3.1 Proposition])**.** *Assume that $\rho(C) = s \cdot \mathrm{Id}$ and $\sigma(C) = r \cdot \mathrm{Id}$. Then*

- *If $r \neq s$ then $H^*(\mathfrak{g}, K; V) = 0$.*
- *If $r = s$ then $H^*(\mathfrak{g}, K; V) = \mathrm{Hom}_K(\wedge^q \mathfrak{p}, V)$.*

Proof. The proof follows these steps:

(1) One defines an inner product on

$$\mathcal{D}^q(V) := \mathrm{Hom}_{\mathbb{R}}(\wedge^q \mathfrak{p}, V) = (\wedge^q \mathfrak{p}) \otimes H \otimes E$$

(observe that we are just taking $\mathbb{R}$-linear homomorphisms and hence $\mathcal{D}^q(V)$ is bigger than $\mathcal{C}^q(V)$) by taking the tensor products of the inner products on each term, which we call $(-,-)_V$. We can then define an adjoint $\partial : \mathcal{D}^q \to \mathcal{D}^{q-1}$ of d for the inner product $(-,-)_V$ and shows the that

$$\Delta := \partial d + d\partial$$

acts on $\mathcal{C}^q(\mathfrak{g}, K; V)$ as $(\rho(C) - \sigma(C)) \cdot \mathrm{Id}$ and that if $\Delta = 0$ then $d = \partial = 0$ (the first assertion follows from a direct calculation and the last one follows from the non-degeneracy of the bilinear pairing).

(2) If $r \neq s$ then for $\eta \in \mathcal{C}^q(\mathfrak{g}, K; V)$ a q-cocycle, we have

$$\Delta\eta = d\partial\eta + \partial d\eta = d\partial\eta$$

and so $\eta = (r - s)^{-1} d\partial\eta$ is a coboundary.

(3) If $r = s$ then $\Delta = 0$ and so $d = 0$ and hence every chain is closed, which gives

$$H^q(\mathfrak{g}, K; V) = \mathrm{Hom}_K(\wedge^q \mathfrak{p}, V).$$

□

Corollary 2. *For trivial coefficients, we can identify*

$$(\wedge^q \mathfrak{p})^K$$

with the G-invariant differential forms on G/K. The result says that all such forms are harmonic, recovering an old result of Cartan

[2] We switch notation and denote the representation V from the previous sections by E.

Corollary 3. *The representations π contributing to the some in Proposition 3 are such that $\chi_\pi = \chi_{\rho^*}$ and $\omega_\pi = \omega_{\rho^*}$, where ρ^* denotes the contragredient representation of ρ and χ_π (resp. χ_{ρ^*}) and ω_π (resp. ω_{ρ^*}) denote the infinitesimal and central characters of π (resp. ρ^*).*

4. Cohomology of tempered representations

Reference: [BW00, §III].

In this section, we calculate the dimension of the $(\mathfrak{g}, K)$-cohomology groups $H^*(\mathfrak{g}, K; V)$ for certain representations $V = H \otimes E$. In particular, we will see that they vanish outside a certain range which is given in purely in terms of G and K.

4.1. Parabolic induction.

Let's start with a definition.

Definition 2. *A parabolic pair is a pair (P, A) where P is a parabolic subgroup and A is a split component of a maximal torus in the Levi of P. Say $(P, A) < (P', A')$ if $P \subset P'$ and $A \supset A'$ and we fix a minimal parabolic pair (P_0, A_0). We say that a parabolic (P, A) is standard (w.r.t. the the chosen minimal parabolic pair) if it is greater than the minimal one.*

Let (P, A) be a standard parabolic pair. Recall the Levi decomposition

$$P = MN = A^0MN.$$

Let (σ, H_σ) be an admissible representations of M with infinitesimal character χ_σ and let $\nu \in \mathfrak{a}_{\mathbb{C}}^*$. Given this data, we define the parabolically induced representation as follows:

$$\begin{aligned} I_{P,\sigma,\nu} &= \operatorname{Ind}_P^G(H_\sigma \otimes \mathbb{C}_{\rho_p+\nu}) \\ &= \{f \in \mathcal{C}^\infty(G, H_\sigma)\ f(man \cdot g) = a^{\rho_p+\nu}\sigma(n)f(g)\}, \end{aligned}$$

where $\rho_P \in \mathfrak{a}_{\mathbb{C}}^*$ is a usual normalisation factor defined as $\rho_P(a) = \det(\operatorname{Ad} a|_{\mathfrak{n}_P})^{1/2}$. This has an action of G by right translation which makes it into an admissible representations of G, which is unitary if $\sigma \otimes \nu$ is unitary, with infinitesimal character $\chi_{\rho_\sigma+\nu}$.

Definition 3. *We call (P, A) cuspidal if ${}^0M$ has a compact Cartan subgroup.*

4.2. Cohomology of induced representations.

Proposition 5 ([BW00, III.5.1 Theorem])**.** *Let (P, A) be a standard cuspidal parabolic pair of G. Let (σ, H_σ) be a discrete series representations of ${}^0M$, $v \in \mathfrak{a}_{\mathbb{C}}^*$ purely imaginary and $I = I_{P,\sigma,v}$. Finally let E be an irreducible and finite dimensional complex representation of G. Then*

(1) $H^q(\mathfrak{g}, K; I \otimes E) = 0$ if $q \notin [q_0, q_0 + l_0]$

(2) If $H^q(\mathfrak{g}, K; I \otimes E) \neq 0$ then it has dimension

$$\binom{\ell_0}{q - q_0}.$$

Recall that the invariants q_0, ℓ_0 are defined as $\ell_0 = \ell_0(G) = \operatorname{rk}(G) - \operatorname{rk}(K)$ and that

$$q_0 = q_0(G) = \frac{\dim(G/K) - \ell_0}{2}$$

Sketch of proof. This is a very deep result with a very involved proof. We content ourselves with sketching the main steps of its proof, unfortunately omitting way too many details.

(1) First one proves (cf. [BW00, §III.2.5]) a version Shapiro's Lemma and obtains
$$H^q(\mathfrak{g}, K; I \otimes E) = H^q(\mathfrak{p}, K_p; H_{\sigma,\nu} \otimes E),$$
where $H_{\sigma,\nu} = H_\sigma \otimes \mathbb{C}_{\rho_p+\nu}$.

(2) There is a Hochschild-Serre spectral sequence (cf. [BW00, §I.6.5]) which reads
$$E_2^{p,q} := H^p\big(\mathfrak{m}, K_P; H^q(\mathfrak{n}, K_N; E) \otimes H_{\sigma,V}\big) \implies H^{p+q}(\mathfrak{p}, K_p; H_{\sigma,\nu} \otimes E).$$

(3) One needs now to understand the groups $H^q(\mathfrak{n}, K_N; E)$ as $\mathfrak{m}$-representations. This is a theorem of Kostant ([BW00, III.3.1 Theorem]):
$$H^q(\mathfrak{n}, E) = \bigoplus_{\substack{s\in W_P\\ l(s)=q}} L_s$$
where W_P is a system of representatives of $W_M\backslash W_G$ (the quotient of the Weyl groups), and the L_s are certain representations which depend on s and on the maximal weight λ of E. Using this one shows that
$$H^{q+l(s)}(\mathfrak{g}, K; I \otimes E) = \left(H^*(\mathfrak{m}, K_P; H_\sigma \otimes L_s) \otimes \wedge^* \mathfrak{a}_{\mathbb{C}}^*\right)^q .$$
The first factor of the RHS is concentrated in degree $q({}^0M) := (\dim {}^0M - \dim K \cap {}^0M)/2$ and has dimension 1 ([BW00, §II.5.4 and II.s.7]). Observe also that $\mathfrak{a}_{\mathbb{C}}^*$ has dimension ℓ_0. This proves that
$$H^{q+l(s)}(\mathfrak{g}, K, I \otimes E) = \wedge^j \mathfrak{a}_{\mathbb{C}}^*$$
where $j = q - q({}^0M)$. This already shows that the dimensions of the cohomology groups are given by some combinatorial numbers and one needs to check that the non-vanishing range is the one claimed in the statement of the Proposition. Finally one shoes that in fact $l(s) = \frac{\dim N}{2}$ and that moreover
$$q_0(G) = q({}^0M) + \frac{\dim N}{2},$$
and the result follows.

□

References

[BW00] A. Borel and N. Wallach. *Continuous cohomology, discrete subgroups, and representations of reductive groups*. Second. Vol. 67. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2000, pp. xviii+260. ISBN: 0-8218-0851-6. DOI: 10.1090/surv/067. URL: https://doi.org/10.1090/surv/067.