DERIVED HECKE ALGEBRAS

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Data

Let G be a reductive group over a number field F. Let $U \subset G(\mathbb{A}_{F,f})$ be an open compact subgroup. Let

$$Y(U) := G(F)\backslash S_\infty \times G(\mathbb{A}_{F,f})/U,$$

where S_∞ is some (possibly disconnected) symmetric space for $G(F \otimes_{\mathbb{Q}} \mathbb{R})$.

There will be some assumptions on these choices later! But the assumptions are not necessary in order to make the basic construction.

1. A toy example of a derived Hecke operator

We consider the following particular case:

- $G = \mathrm{Res}_{F/\mathbb{Q}} \mathrm{PGL}_2$, where $F/\mathbb{Q}$ is quadratic imaginary with class number 1. Let $\mathcal{O} \subset F$ denote its ring of integers.
- Let $Y(1)$ be the full level quotient, that is,
$$Y(1) := \mathrm{PGL}_2(\mathcal{O})\backslash \mathbb{H}^3.$$
- $Y_0(q) = Y(\Gamma_0(q)) = \Gamma_0(q)\backslash \mathbb{H}^3$, where
$$\Gamma_0(q) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mid c \in q\}/\sim,$$
for a prime ideal $q \subset \mathcal{O}$.

Fix $\alpha : \mathbb{F}_q^\times \to \mathbb{Z}/\ell^n\mathbb{Z}$. This gives rise to a cohomology class $\langle \alpha \rangle \in H^1(\Gamma_0(q), \mathbb{Z}/\ell^n\mathbb{Z})$, via

$$\Gamma_0(q) \to \mathbb{F}_q^\times \xrightarrow{\alpha} \mathbb{Z}/\ell^n\mathbb{Z}, \qquad \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \frac{a}{d}.$$

The usual Hecke operator for q arises from a correspondence

$$\begin{tikzcd} & H^*\Gamma_0(q) \arrow[dr, "\pi_{2*}"] & \\ H^*\mathrm{PGL}_2(\mathcal{O}) \arrow[ur, "\pi_1^*"] \arrow[rr, "T_q"] & & H^*\mathrm{PGL}_2(\mathcal{O}) \end{tikzcd}$$

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For the cohomology with $\mathbb{Z}/\ell^n\mathbb{Z}-$coefficients, we can construct a derived version of the Hecke operator via the diagram

$$\begin{tikzcd}
& H^*\Gamma_0(q) \arrow[r, "\cup\langle\alpha\rangle"] & H^{*+1}\Gamma_0(q) \arrow[dr, "\pi_{2*}"] & \\
H^*\mathrm{PGL}_2(\mathcal{O}) \arrow[ur, "\pi_1^*"] \arrow[rrr, "T_{q,\langle\alpha\rangle}"] & & & H^{*+1}\mathrm{PGL}_2(\mathcal{O})
\end{tikzcd}$$

simply by composing with the cup-product with the class $\langle\alpha\rangle$.

Remark 1.1. This construction truly arises in finite characteristic. To obtain a larger class of derived hecke operators, we require

$$\ell^n \mid |\mathbb{F}_q^\times| = N(q) - 1.$$

If we want a characteristic zero action, we need limits, such as

$$\begin{tikzcd}
H^*(\mathrm{PGL}_2(\mathcal{O}), \mathbb{Z}/\ell^{n+1}\mathbb{Z}) \arrow[r, "T_{q_{n+1}},\langle\alpha_{n+1}\rangle"] \arrow[d] & H^{*+1}(\mathrm{PGL}_2(\mathcal{O}), \mathbb{Z}/\ell^{n+1}\mathbb{Z}) \arrow[d] \\
H^*(\mathrm{PGL}_2(\mathcal{O}), \mathbb{Z}/\ell^{n}\mathbb{Z}) \arrow[r, "T_{q_n},\langle\alpha_n\rangle"] & H^{*+1}(\mathrm{PGL}_2(\mathcal{O}), \mathbb{Z}/\ell^{n}\mathbb{Z})
\end{tikzcd}$$

over a system of Taylor–Wiles primes, where α_n arises from a prime q_n such that $\ell^n \mid (N(q_n) - 1)$ for all $n \geq 1$.

Remark 1.2. The class $\langle\alpha\rangle$ is a *congruence class*, meaning that it vanishes on a congruence subgroup; nevertheless, this gives rise to non-trivial operators!

2. Definitions of derived Hecke algebras

2.1. **Homological.** This is a rather abstract definition. Start with these data.

- v is a place of F, with residue field $\mathbb{F}_v$, of characteristic p
- S is a coefficient ring, with characteristic ℓ, where $v \nmid \ell$

The goal is constructing a "ℓ-adic derived Hecke algebra of a v-adic group", working locally with

$$G = G(F_v), \quad U = U_v.$$

The derived Hecke algebra

$$\mathcal{H}(G, U)$$

is a graded algebra such that its degree 0-graded piece is the usual Hecke algebra. Thus, let

$$\mathcal{H}(G, U)^{(0)} = \mathrm{Hom}_{SG}(S[G/U], S[G/U]),$$

where $S[X]$ is a free module over a set X, and SG is the category of smooth G-representations with coefficients in S. This is one possible characterization of the usual Hecke algebra. Its derived extension is defined as

$$\mathcal{H}(G, U) := \mathrm{Ext}^*_{SG}(S[G/U], S[G/U]).$$

More concretely, to produce this one takes a projective resolution

$$P^\bullet \longrightarrow S[G/U],$$

and then $\mathcal{H}(G, U)$ is the cohomology of the differential graded algebra

$$\underline{\mathrm{Hom}}(P^\bullet, P^\bullet).$$

2.2. **Derived Hecke algebras via invariant functions.** This may seem quite different from the homological definition.

Let $(x, y) \in G/U \times G/U$; we can write $x = g_x U, y = g_y U$, for some representatives $g_x, g_y \in G$. Let $G_{x,y}$ be the (pointwise) stabilizer of (x, y). An element of the derived Hecke algebra $\mathcal{H}(G, U)$ is a function assigning to every (x, y) a class

$$h(x, y) \in H^*(G_{x,y}, S)$$

such that

(i) it satisfies the G-invariance condition

$$h(x, y) = [g]^* h(gx, gy),$$

where $[g] = \mathrm{Ad}(g) : G_{x,y} \to G_{gx,gy}$

(ii) it satisfies the G-finiteness condition: there exists a finite subset $T \subset G/U \times G/U$ such that

$$h(x, y) \text{ is supported on } GT.$$

The addition operation is clear; multiplication is more subtle. For $h_1, h_2 \in \mathcal{H}(G, U)$, we have

$$(h_1 * h_2)(x, z) = \sum_{y \in G/U} h_1(x, y) \cup h_2(y, z),$$

where $h_1(x, y) \in H^*(G_{x,y}, S)$ while $h_2(y, z) \in H^*(G_{y,z}, S)$: this cup product makes sense in the following way:

- view $h_1(x, y), h_2(y, z)$ as elements of $H^*(G_{x,y,z}, S)$ (here $G_{x,y,z}$ is the stabilizer of (x, y, z) under the diagonal action), so $h_1(x, y) \cup h_2(y, z)$ is defined here.
- then, rewrite $(h_1 * h_2)(x, z)$ as

$$\begin{aligned}(h_1 * h_2)(x, z) &= \sum_{y_0:\text{reps of } G_{xz}\backslash G/U} \sum_{y \in O(y_0)} h_1(x, y) \cup h_2(y, z) \\ &= \sum_{y_0:\text{reps of } G_{x,z}\backslash G/U} \mathrm{cores}_{G_{x,z}}^{G_{x,y_0,z}} h(x, y_0) \cup h_2(y_0, z).\end{aligned}$$

(here $O(y_0)$ denotes the orbit of y_0).

Here is a variant of this definition: consider

$$\bigoplus_{z \in U\backslash G/U} H^*(U_z, S),$$

where U_z is the stabilizer of $z \in U\backslash G$ in U; then U_z which is also equal to the stabilizer in G of the pair (z, e) where $e = 1_G U$, so $U_z = G_{(z,e)}$. Then, we have a natural map

$$\bigoplus_{z \in U\backslash G/U} H^*(U_z, S) \longrightarrow \mathcal{H}(G, U).$$

Let α be an element of one of these summands over a given z. Now, define $h = h(\alpha, z)$ as

- $h = 0$ outside the G-orbit of (z, e) in $G/U \times G/U$
- $h(z, e) = \alpha$ in $H^*(G_{z,e}, S) = H^*(U_z, S)$
- we extend h by G-invariance to the orbit of (z, e)

The description of multiplication is not very natural in this definition.

Example 2.1. For $G = \mathrm{GL}_2(F_v)$, $U = \mathrm{GL}_2(\mathcal{O}_v)$, and $\pi \in \mathcal{O}_v$ a uniformizer, there is the Cartan decomposition

$$G = \coprod_{a \leq b \in \mathbb{Z}} U \begin{pmatrix} \pi^b & 0 \\ 0 & \pi^a \end{pmatrix} U.$$

Let $U^{a,b}$ be defined by

$$U^{a,b} = U_{\left(\begin{smallmatrix} \pi^b & 0 \\ 0 & \pi^a \end{smallmatrix}\right)},$$

the stabilizer of $\left(\begin{smallmatrix} \pi^b & 0 \\ 0 & \pi^a \end{smallmatrix}\right)$ in U. This is

$$\begin{aligned} U^{a,b} &= U \cap \mathrm{Ad}(\begin{pmatrix} \pi^b & 0 \\ 0 & \pi^a \end{pmatrix}) \cdot U \\ &= \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : \pi^{b-a} \mid \gamma \right\}. \end{aligned}$$

Then

$$H^*(U^{a,b}, S) = H^*(U^{a,b}/U^{a,b,(p)}, S),$$

where $U^{a,b,(p)}$ denotes the maximal pro p-quotient of $U^{a,b}$. For $b > a$, is equal to

$$H^*(\mathbb{G}_m(\mathbb{F}_v) \times \mathbb{G}_m(\mathbb{F}_v), S),$$

where $\mathbb{F}_v$ is the residue field. The order of the group is $(q_v - 1)^2$, where $q_v := |\mathbb{F}_v|$.

This derived Hecke algebra is more interesting when $q_v - 1 = 0$ is in S.

2.3. **Comparison of the two definitions.** In the first definition, we want to calculate the cohomology of the dg-algebra

$$\underline{\mathrm{Hom}}_{SG}(P^\bullet, P^\bullet),$$

where $P^\bullet \to S[G/U]$ is a projective resolution. In order to construct a nice choice of resolution, we start with

$$Q^\bullet \to S,$$

a projective resolution of S in the category of SU-modules. This is rather straightforward, using the structure of the groups. Then, from this, we induce (compact induction, termwise):

$$P^\bullet := \mathrm{Ind}_U^G(Q^\bullet) \longrightarrow \mathrm{Ind}_U^G S = S[G/U].$$

It will be important to understand $\mathrm{Ind}_U^G(Q^\bullet)$ as a U-module. We get

$$\mathrm{Ind}_U^G(Q^\bullet) \cong \bigoplus_{x \in U \backslash G/U} \mathrm{Ind}_{U_x}^U(Q_x),$$

where $Q_x = Q$ as an S-module, with U_x-action given by $u \mapsto \mathrm{Ad}(x) \cdot u$.

Then we write

$$\begin{aligned} \underline{\mathrm{Hom}}_{SG}(P^\bullet, P^\bullet) &= \underline{\mathrm{Hom}}_{SG}(\mathrm{Ind}_U^G(Q^\bullet), \mathrm{Ind}_U^G(Q^\bullet)) \\ &= \underline{\mathrm{Hom}}_{SU}(Q^\bullet, \mathrm{Ind}_U^G(Q^\bullet)) \\ &= \underline{\mathrm{Hom}}_{SU}(Q^\bullet, \bigoplus_{x \in U \backslash G/U} \mathrm{Ind}_{U_x}^U(Q_x)), \end{aligned}$$

where the second equality comes from Frobenius reciprocity. The last term receives a map from

$$\bigoplus_{x \in U \backslash G/U} \underline{\mathrm{Hom}}_{SU}(Q^\bullet, \mathrm{Ind}_{U_x}^U(Q_x^\bullet)),$$

which commutes with taking cohomology. Then we calculate the terms of the RHS:

$$\underline{\mathrm{Hom}}_{SU}(Q^\bullet, \mathrm{Ind}_{U_x}^{U}(Q_x^\bullet)) = \underline{\mathrm{Hom}}_{SU_x}(Q^\bullet, Q_x^\bullet).$$

Since $Q^\bullet$ and $Q_x^\bullet$ are resolutions of S, the cohomology of this complex is $H^*(U_x, S)$.

Its not too hard to see that this induces an additive map between the two definitions, but checking that it respects multiplication is harder (but true).

3. Action of the derived Hecke algebra on $H^*(Y(U), S)$

3.1. **Homological version of the action.** We now return to the global setting. Let $U \subset G(\mathbb{A}_f)$ be an open compact subgroup. The (local) homological definition is

$$\mathcal{H}(G_v, U_v) := H^*(\underline{\mathrm{Hom}}_{SG_v}(P^\bullet, P^\bullet)).$$

This acts on the cohomology of

$$\underline{\mathrm{Hom}}_{SG_v}(S[G_v/U_v], M^\bullet),$$

as $S[G_v/U_v]$ may be replaced by $P^\bullet$ for the purpose of computing cohomology.

For $V_v \subset G(F_v)$ an open compact subgroup, we define the cochains for this subgroup as

$$\mathcal{C}^\bullet(V_v) := \mathcal{C}^\bullet(Y(U^{(v)} \times V_v), S).$$

Let

$$M^\bullet := \varinjlim_{V_v \subset G_v} \mathcal{C}^\bullet(V_v).$$

where V_v varies over all open compact subgroups of G_v.

Proposition 3.1. *The cohomology of* $\underline{\mathrm{Hom}}_{SG_v}(S[G_v/U_v], M^\bullet)$ *is* $H^*(Y(U), S)$.

This gives an action the derived Hecke algebra on the cohomology of $Y(U)$.

3.2. **Practical version of the action.** We would like to have a map

$$H^*(U_v, S) \longrightarrow H^*(Y(U), S),$$

which we would apply to a class $\alpha \in H^1(U_v/U_{v,1}, S)$. Here is how we produce it: define $U_1 \subset U$ as the preimage of $U_{v,1} \subset U_v$ under the natural map $U \to U_v$. Thus we have a map

$$Y(U_1) \longrightarrow Y(U),$$

which is a cover with transformation group $U_v/U_{v,1}$. Thus we get a map from $Y(U)$ to the classifying space of $U_v/U_{v,1}$, reflecting this $U_v/U_{v,1}$-torsor. This induces, in particular,

$$H^*(U_v/U_{v,1}, S) \longrightarrow H^*(Y(U), S).$$

We let $\langle\alpha\rangle \in H^*(Y(U), S)$, a "congruence class" as discussed at the outset, represent the image of $\alpha \in H^*(U_v/U_{v,1}, S)$.

What is the action of $h_{z,\alpha}$ on $H^*(Y(U), S)$? Well, U_z is the stabilizer of $z = g_zU$ in U. Choose also $U_{z'}$, the stabilizer of $g_z^{-1}U$. The we have a concrete description of the action:

$$\begin{tikzcd}
& H^*(Y(U_z)) \arrow[r, "\cup\langle\alpha\rangle"] & H^*(Y(U_z)) \arrow[r, "f_*"] & H^*(Y(U_{z'})) \arrow[dr, "\pi_{2,*}"] & \\
H^*(Y(U)) \arrow[ur, "\pi_1^*"] \arrow[rrrr, "h_{z,\alpha}"] & & & & H^{*+1}(Y(U))
\end{tikzcd}$$

where f^* arises from $f : g \mapsto gg_z$.