DERIVED SATAKE ISOMORPHISM AND IWAHORI-HECKE ALGEBRAS, [?, §3-4]

Last week in the Hecke track, Alice introduced a local derived Hecke algebra and change of degree action on the cohomology of a locally symmetric space. However, there is lots still to do: we do not even know whether this action is non-zero yet! The goals of this talk are:

(1) (Derived Satake) Understand the structure of the local derived Hecke algebra $\mathscr{H}(\mathrm{G},\mathrm{K})$ with K a *hyperspecial* maximal compact subgroup as an algebra (last week, we understood an explicit decomposition as an S-module) and this is the subject of §3 of Venkatesh. That is the main goal of today.
(2) (Derived I–H) Introduce some closely related Iwahori–Hecke algebras and their derived analogues, §4 of Venkatesh.

1. Structure of split reductive p-adic groups

These two sections are purely local, let's review some notation: $\mathrm{F}/\mathbb{Q}_p, \mathscr{O}, \mathrm{k} = \mathbf{F}, \sharp\mathrm{k} = q, \varpi$ As this is a purely local talk, we have dropped all ν's.

$\mathbf{G}$ F-split connected reductive group/F, set $\mathrm{G} = \mathbf{G}(\mathrm{F})$;

$\mathbf{A}$ a maximal torus in $\mathbf{G}$;

K a hyperspecial maximal compact open subgroup in the apartment of $\mathbf{A}$;

$\mathcal{G}/\mathscr{O}$ smooth integral model of $\mathbf{G}$, (generic fibre $\mathbf{G}$)
with $\mathrm{K} = \mathcal{G}(\mathscr{O})$ and connected reductive special fibre;

$\mathcal{B} = \mathcal{A} \ltimes \mathcal{N}$ Borel subgroup and Levi decomposition $\mathcal{G}$ with, $\mathbf{B}, \mathbf{A}, \mathbf{N}$ generic fibres;

$\mathrm{B} = \mathbf{B}(\mathrm{F}),\ \mathrm{A} = \mathbf{A}(\mathrm{F}),\ \mathrm{N} = \mathbf{N}(\mathrm{F})$;

$\underline{\mathrm{G}} = \mathcal{G}(\mathrm{k}),\ \underline{\mathrm{B}} = \mathcal{B}(\mathrm{k}),\ \underline{\mathrm{A}} = \mathcal{A}(\mathrm{k})(= \mathrm{T}$ in Venkatesh);

$\mathrm{I} \leqslant \mathrm{K}$ an *Iwahori* subgroup of G chosen relative to $\underline{\mathrm{B}}$;

$\mathrm{A} \cap \mathrm{K}$ is the unique maximal compact subgroup of A;

$\mathrm{W} = \mathrm{N}_{\mathbf{G}}(\mathbf{A})/\mathbf{A}$ Weyl group;

$\mathrm{X}_* = \mathrm{X}_*(\mathbf{A}) = \mathrm{Hom}(\mathbb{G}_m, \mathbf{A})$;

$\mathrm{X}^* = \mathrm{X}^*(\mathbf{A}) = \mathrm{Hom}(\mathbf{A}, \mathbb{G}_m)$;

$\Phi^+ \subset \mathrm{X}^*$ characters appearing in the adjoint representation on the Lie algebra of $\mathbf{B}$;

$$\mathrm{X}_*^+ = \{\chi \in \mathrm{X}_* : \langle \chi, \psi \rangle \geqslant 0 \text{ for all } \psi \in \Phi^+\};$$
$$\mathrm{X}_* \xrightarrow{\simeq} (\mathrm{A}/\mathrm{A} \cap \mathrm{K}),\ \chi \mapsto \chi(\varpi) \text{ (identify)};$$
$$\widetilde{\mathrm{W}} = \mathrm{X}_* \rtimes \mathrm{W} \text{ affine Weyl group;}$$

(Cartan Decomposition): $\mathrm{G} = \mathrm{KAK} = \bigsqcup_{\chi \in \mathrm{X}_*^+} \mathrm{K}\chi(\varpi)\mathrm{K}$

(The $\mathrm{A}/\mathrm{K} \cap \mathrm{A}$ component in the Cartan decomposition is unique up to the action of W.)

(Iwasawa Decomposition): $\mathrm{G} = \mathrm{BK}$;

(Bruhat Decomposition): $\mathrm{G} = \mathrm{I}\widetilde{\mathrm{W}}\mathrm{I}$.

For example (put $\mathrm{GL}_2(\mathbb{Q}_p)$ example running down the side above)

$$\mathbf{G} = \mathrm{GL}_2/\mathbb{Q}_p$$
$$\mathrm{G} = \mathrm{GL}_2(\mathbb{Q}_p),\ \mathrm{B} = \left(\begin{smallmatrix} \star & \star \\ 0 & \star \end{smallmatrix}\right) \cap \mathrm{G}, \mathrm{A} = \left(\begin{smallmatrix} \star & 0 \\ 0 & \star \end{smallmatrix}\right) \cap \mathrm{G},$$
$$\mathrm{K} = \mathrm{GL}_2(\mathbb{Z}_p);$$
$$\underline{\mathrm{G}} = \mathrm{GL}_2(\mathbb{F}_p), \qquad \underline{\mathrm{B}} = \left(\begin{smallmatrix} \star & \star \\ 0 & \star \end{smallmatrix}\right) \cap \mathrm{GL}_2(\mathbb{F}_p), \text{etc;}$$
$$\mathrm{I} = \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}^\times$$
$$\mathrm{W} = \mathrm{S}_2, \text{ represented by monomial matrices } \{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\}$$
$$\mathrm{X}_* = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2,\ e_1(x) = \mathrm{diag}(x,1),\ e_2(x) = \mathrm{diag}(1,x);$$
$$\mathrm{X}^* = \mathbb{Z}\chi_1 \oplus \mathbb{Z}\chi_2,\ \chi_1(\mathrm{diag}(x,y)) = x,\ \chi_2(\mathrm{diag}(x,y)) = y;$$
$$\Phi^+ = \{\chi_1 - \chi_2\} \subset \Phi = \{\chi_1 - \chi_2, \chi_2 - \chi_1\};$$
$$\mathrm{X}_*^+ = \{ae_1 + be_2 : a \geqslant b\};$$
$$\widetilde{\mathrm{W}} = \left\{ \begin{pmatrix} p^a & 0 \\ 0 & p^b \end{pmatrix}, \begin{pmatrix} 0 & p^a \\ p^b & 0 \end{pmatrix} \right\}$$
$$\mathrm{G} = \bigsqcup_{a \geqslant b} \mathrm{K} \begin{pmatrix} p^a & 0 \\ 0 & p^b \end{pmatrix} \mathrm{K} \text{ (refined) Cartan decomposition.}$$

2. Classical Satake/"Degree 0"

For now, S denotes a commutative ring.

Define the K-*spherical Hecke algebra* by

$$\mathrm{H}_\mathrm{K} = \mathrm{End}_{\mathrm{S}[\mathrm{G}]}(\mathrm{S}[\mathrm{G}/\mathrm{K}]),$$

under composition of endomorphisms. Using Frobenius reciprocity

$$\mathrm{H}_\mathrm{K} \simeq \{f : \mathrm{G} \to \mathrm{S} \text{ compactly supported, left and right K-invariant}\},$$

an algebra under convolution of functions

$$f \star f'(g) = \sum_{x \in \mathrm{G}/\mathrm{K}} f(x) f'(x^{-1}g).$$

Example 1. $(\mathrm{A}, \mathrm{A} \cap \mathrm{K})$ is a special case, and

$$\mathrm{H}_{\mathrm{A}\cap\mathrm{K}} \simeq \mathrm{S}[\mathrm{A}/\mathrm{A} \cap \mathrm{K}] = \mathrm{S}[\mathrm{X}_*].$$

And $\mathrm{X}_* \simeq \mathbb{Z}^r$, for some r, and $\mathrm{H}_{\mathrm{A}\cap\mathrm{K}} \simeq \mathrm{S}[\mathrm{T}_1^{\pm 1}, \cdots, \mathrm{T}_r^{\pm 1}]$ is commutative.

Let dn be a complex valued *Haar measure* on N normalised by $dn(\mathrm{N}\cap\mathrm{K}) = 1$, and $\delta_{\mathrm{B}} : \mathrm{A} \to \mathbb{C}^\times$ the *modulus character* defined by

$$\int_{\mathrm{N}} f(a^{-1}na)dn = \delta_{\mathrm{B}}(a) \int_{\mathrm{N}} f(n)dn.$$

Theorem 2 (Satake over $\mathbb{C}$)**.** With $\mathrm{S} = \mathbb{C}$, the map

$$\begin{aligned} \mathcal{S} : \mathrm{H}_{\mathrm{K}} &\to \mathrm{H}_{\mathrm{A}\cap\mathrm{K}} \\ f &\mapsto \mathrm{S}(f)(a) = \delta_{\mathrm{B}}(a)^{1/2} \int_{\mathrm{N}} f(an)dn \end{aligned}$$

is an injective algebra homomorphism, with image equal to the W-invariant functions in $\mathrm{H}_{\mathrm{A}\cap\mathrm{K}}$. In particular, H_{K} is commutative.

One uses the Cartan decomposition to show $\mathcal{S}$ is bijective onto the W-invariants, and the Iwasawa decomposition to show $\mathcal{S}$ is an algebra homomorphism.

Example 3. Let $\mathrm{T}_1 = 1_{\mathrm{K}\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)\mathrm{K}}$, $\mathrm{T}_2 = 1_{\mathrm{K}\left(\begin{smallmatrix} p & 0 \\ 0 & p \end{smallmatrix}\right)\mathrm{K}}$, then

$$\begin{aligned} \mathcal{S}(\mathrm{T}_2) &= 1_{\left(\begin{smallmatrix} p & 0 \\ 0 & p \end{smallmatrix}\right)(\mathrm{A}\cap\mathrm{K})} \\ \mathcal{S}(\mathrm{T}_1) &= q^{1/2}\left(1_{\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)(\mathrm{A}\cap\mathrm{K})} + 1_{\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)(\mathrm{A}\cap\mathrm{K})}\right) \end{aligned}$$

and $\mathrm{H}_{\mathrm{K}} \simeq \mathbb{C}[\mathrm{T}_1, \mathrm{T}_2, \mathrm{T}_2^{-1}]$. *Notice that replacing q with 1, for a choice of square root, $\mathcal{S}$ coincides with restriction!*

Remark 4. Let (π, V) be a smooth irreducible representation of G over $\mathbb{C}$. Then H_{K} acts on V^{K} and if it is non-zero it is a simple H_{K}-module, and the map

$$\begin{aligned} &\text{irreducible smooth representations of G with non-zero K-invariants}/\simeq \\ &\qquad \to \text{simple } \mathrm{H}_{\mathrm{K}}\text{-modules}/\simeq, \end{aligned}$$

is a bijection. *Using the Satake isomorphism and introducing the dual group into the picture, Langlands rewrote this as a parametrisation of these irreducible representations with* K*-invariants by semisimple conjugacy classes in* ${}^L\mathrm{G}$.

3. Venkatesh's Derived Version

From now on $\mathrm{S} = \mathbb{Z}/\ell^r$, $\ell \neq p$ (we could work in greater generality, but to follow last week, we at least for the moment need to ensure our category of smooth representations has enough projectives and injectives).

Last week, we introduced a derived Hecke algebra with coefficients in S

$$\mathscr{H}_{\mathrm{K}} = \mathscr{H}(\mathrm{G}, \mathrm{K}) = \mathrm{Ext}^*(\mathrm{S}[\mathrm{G}/\mathrm{K}], \mathrm{S}[\mathrm{G}/\mathrm{K}]),$$

where Ext is taken in the category of smooth G-modules. And gave a description of $\mathscr{H}_{\mathrm{K}}$ as an algebra of functions

$$h : \mathrm{G}/\mathrm{K} \times \mathrm{G}/\mathrm{K} \to \bigoplus_{(x,y)\in \mathrm{G}/\mathrm{K}} \mathrm{H}^*(\mathrm{G}_{xy}, \mathrm{S}),$$

where $\mathrm{G}_{xy} = \mathrm{Stab}_{\mathrm{G}}(x, y) = (x\mathrm{K}x^{-1}) \cap (y\mathrm{K}y^{-1})$ is the pointwise stabilizer of (x, y) in G, which satisfy:

(0) $h(x, y) \in \mathrm{H}^*(\mathrm{G}_{xy}, \mathrm{S})$;
(1) h is G-invariant;
 ($[g]^*h(gx, gy) = h(x, y)$ where $[g]^* : \mathrm{H}^*(\mathrm{G}_{gxgy}, \mathrm{S}) \to \mathrm{H}^*(\mathrm{G}_{xy}, \mathrm{S})$ is induced from $\mathrm{Ad}(g)$.)
(2) h has finite support modulo G;
 (there exists a finite subset $\mathrm{R} \subset \mathrm{G}/\mathrm{K} \times \mathrm{G}/\mathrm{K}$ such that $h(x, y) = 0$ if (x, y) is not in the G-orbit of R.)

under a "convolution" product.

$$h_1 * h_2(x, z) = \sum_{y\in \mathrm{G}/\mathrm{K}} h_1(x, y) \cup h_2(y, z) \text{ interpreted suitably in } \mathrm{H}^*(\mathrm{G}_{xz}, \mathrm{S}).$$

We also gave a description of $\mathscr{H}_{\mathrm{K}}$ as an S-module using the Cartan decomposition which we will use later.

Example 5. Simple example (A, $\mathrm{K} \cap \mathrm{A}$): For all $(x, y) \in (\mathrm{A}/\mathrm{K} \cap \mathrm{A})^2$, since A is abelian, $\mathrm{A}_{xy} = \mathrm{A} \cap \mathrm{K}$! Hence we are looking at functions

$$h : (\mathrm{A}/\mathrm{K} \cap \mathrm{A}) \times (\mathrm{A}/\mathrm{K} \cap \mathrm{A}) \to \mathrm{H}^*(\mathrm{A} \cap \mathrm{K}, \mathrm{S})$$
$$h : \mathrm{X}_* \times \mathrm{X}_* \to \mathrm{H}^*(\mathrm{A} \cap \mathrm{K}, \mathrm{S})$$

constant on A-orbits, of finite support, which thus identifies with

$$\mathrm{S}[\mathrm{A}/\mathrm{K} \cap \mathrm{A}] \otimes_{\mathrm{S}} \mathrm{H}^*(\mathrm{A} \cap \mathrm{K}, \mathrm{S}).$$

with multiplication in the group algebra in the X_*-variable and multiplication in $\mathrm{H}^*(\mathrm{A} \cap \mathrm{K}, \mathrm{S})$ and the grading coming from $\mathrm{H}^*(\mathrm{A} \cap \mathrm{K}, \mathrm{S})$. As such it is graded commutative.

The reduction map $\mathrm{A} \cap \mathrm{K} \to \underline{\mathrm{A}}$ splits uniquely, and under our hypotheses induces an isomorphism on cohomology with coefficients in S

$$\mathrm{H}^*(\mathrm{A} \cap \mathrm{K}, \mathrm{S}) \simeq \mathrm{H}^*(\underline{\mathrm{A}}, \mathrm{S})$$

Following Venkatesh, we make this identification for the remainder.

To summarise, for the derived Hecke algebra of a torus we have taken the spherical Hecke algebra of the torus and tensored it with $\mathrm{H}^*(\underline{\mathrm{A}}, \mathrm{S})$.

Theorem 6 (Derived Satake, 3.3)**.** Let $\mathrm{S} = \mathbb{Z}/\ell^r$, where ℓ is prime, $\ell^r \mid (q-1)$, $\ell \nmid \sharp\mathrm{W}$ (strong assumptions). Then restriction defines an isomorphism of algebras

$$\mathscr{H}_{\mathrm{K}} \xrightarrow{\sim} \mathscr{H}_{\mathrm{A}\cap\mathrm{K}}^{\mathrm{W}}.$$

Let h be as above. Let us explain what restriction means:

$$\begin{array}{ccc} (x,y) \in (\mathrm{G}/\mathrm{K})^2 & \xrightarrow{\quad h \quad} & h(x,y) \in \mathrm{H}^*(\mathrm{G}_{xy}, \mathrm{S}) \\ \uparrow & & \Big\downarrow {\scriptstyle \mathrm{A}_{xy} \hookrightarrow \mathrm{G}_{xy}} \\ (x,y) \in (\mathrm{A}/\mathrm{A}\cap\mathrm{K})^2 & \xrightarrow{\ h' \ } & h'(x,y) \in \mathrm{H}^*(\mathrm{A}_{xy}, \mathrm{S}) \simeq \mathrm{H}^*(\underline{\mathrm{A}}, \mathrm{S}) \end{array}$$

The properties (1)-(2) for h' are straightforward. In addition the G-invariance of h means that h' is also W-invariant.

Corollary 7. Under above assumptions on (ℓ, q).

(1) $\mathscr{H}_{\mathrm{K}}$ is graded commutative.
(2) If $\ell^r \mid (q-1)$, then the induced map

$$\mathscr{H}_{\mathrm{K}} \text{ over } \mathrm{S} = \mathbb{Z}/\ell^r \to \mathscr{H}_{\mathrm{K}} \text{ over } \mathbb{Z}/\ell^s$$

is surjective for all $s < r$.

Theorem 6 reduces Part 2 to the case of a torus, which follows from the fact:, for C a finite cyclic group with $\ell^r \mid \sharp\mathrm{C}$, the map $\mathrm{H}^*(\mathrm{C}, \mathbb{Z}/\ell^r) \to \mathrm{H}^*(\mathrm{C}, \mathbb{Z}/\ell^s)$ is surjective.

Proof of Theorem 3.3. We first show that $h \mapsto h'$ is a bijection. Using the Cartan decomposition we have a decomposition (see last time for the details) as an S-module:

$$\mathscr{H}_{\mathrm{K}} = \bigoplus_{a \in \mathrm{X}_*^+} \mathrm{H}^*(\mathrm{K} \cap \mathrm{Ad}(a)\mathrm{K}, \mathrm{S})$$

We can write $h = \sum h_{a,\alpha}$ uniquely with $a \in \mathrm{X}_*^+$ and $\alpha \in \mathrm{H}^*(\mathrm{K} \cap \mathrm{Ad}(a)\mathrm{K}, \mathrm{S})$ and

$$\mathrm{supp}(h_{a,\alpha}) = \mathrm{G} \cdot (a, e), \qquad h_{a,\alpha}(a,e) = \alpha.$$

We have $\mathrm{K}a\mathrm{K} \cap \mathrm{X}_* = \mathrm{W} \cdot a$. And the element $h'_{a,\alpha}$ is characterized by the following properties:

$$\begin{aligned} &h'_{a,\alpha} \text{ is W-invariant;} \\ &h'_{a,\alpha}(x,e) = 0 \text{ unless } x \in \mathrm{W} \cdot a \\ &h_{a,\alpha}(a,e) = \mathrm{Res}_{\underline{\mathrm{A}}}^{\mathrm{K}\cap\mathrm{Ad}(a)\mathrm{K}}(\alpha). \end{aligned}$$

It is enough, to show each element of $\mathscr{H}(\mathrm{A}, \mathrm{A} \cap \mathrm{K})^{\mathrm{W}}$ is uniquely the sum of elements $h'_{a,\alpha}$. This follows from:

Claim: $\mathrm{Res} : \mathrm{H}^*(\mathrm{K}\cap\mathrm{Ad}(a)\mathrm{K}, \mathrm{S}) \to \mathrm{H}^*(\underline{\mathrm{A}}, \mathrm{S})^{\mathrm{W}_a}$ is an isomorphism, where $\mathrm{W}_a = \mathrm{Stab}_{\mathrm{W}}(a)$. *The group* $\mathrm{K} \cap \mathrm{Ad}(a)\mathrm{K}$ *is modulo its maximal normal pro-p subgroup is equal to the Levi subgroup* $\mathrm{M}(\mathrm{k})$ *of* $\mathcal{G}(\mathrm{k})$ *that centralises a with Weyl group* W_a*. Modulo its maximal pro-p subgroup* $\mathrm{A} \cap \mathrm{K}$ *is equal to torus in* $\mathrm{M}(\mathrm{k})$*. So we can reduce to:*

Lemma 8 (3.7)**.** Under our assumptions on (q, ℓ), the restriction map $\mathrm{H}^*(\underline{\mathrm{G}}, \mathrm{S}) \to \mathrm{H}^*(\underline{\mathrm{A}}, \mathrm{S})^{\mathrm{W}}$ is an isomorphism.

Proof. The restriction map defines an isomorphism $\mathrm{H}^*(\underline{\mathrm{B}}, \mathrm{S}) \xrightarrow{\sim} \mathrm{H}^*(\underline{\mathrm{A}}, \mathrm{S})$ and its inverse is given by corestriction as $\underline{\mathrm{B}}$ is the semidirect product of $\underline{\mathrm{A}}$ with a unipotent q-group of order congruent to 1 modulo ℓ. We transport the W-action on $\mathrm{H}^*(\underline{\mathrm{A}}, \mathrm{S})$ to an action on $\mathrm{H}^*(\underline{\mathrm{B}}, \mathrm{S})$ by

$$\mathrm{Cores} \circ [w]_{\underline{\mathrm{A}}} \circ \mathrm{Res}.$$

We now forget about $\underline{\mathrm{A}}$ now and consider the restriction map

$$\mathrm{H}^*(\underline{\mathrm{G}}, \mathrm{S}) \to \mathrm{H}^*(\underline{\mathrm{B}}, \mathrm{S})$$

We have

$$\begin{aligned} \mathrm{Res}^G_B \circ \mathrm{Cores}^G_B &= \sum_{w\in\mathrm{W}} \mathrm{Cores}^B_{wBw^{-1}\cap B} \mathrm{Ad}(w^{-1})^* \mathrm{Res}^B_{B\cap w^{-1}Bw} \\ &= \sum_{w\in\mathrm{W}} [w]_{\mathrm{B}} \end{aligned}$$

as $\mathrm{Cores}_{\mathrm{A}}^{w\mathrm{B}w^{-1}\cap\mathrm{B}} \circ \mathrm{Res}_{\mathrm{A}}^{w\mathrm{B}w^{-1}\cap\mathrm{B}} = 1$. Morevover, by the Bruhat decomposition under our hypotheses $\mathrm{Cores} \circ \mathrm{Res} = |\mathrm{W}| \in \mathrm{S}^\times$. Hence Res is surjective and Cores is injective, and by the above ResCores computation $\mathrm{Res}^{\mathrm{G}}_{\mathrm{B}}$ has image the W-invariants. □

We now sketch why $h \mapsto h'$ is an algebra homomorphism:

$$\begin{aligned} (h_1h_2)'(x,z) &= \mathrm{Res}_{\underline{\mathrm{A}}}^{G_{xz}} \text{“} \sum_{y\in\mathrm{G}/\mathrm{K}} h_1(x,y) \cup h_2(y,z)\text{”} \\ &= \sum_{\mathrm{G}_{xz}\backslash\mathrm{G}/\mathrm{K}} \mathrm{Res}_{\underline{\mathrm{A}}}^{\mathrm{G}_{xz}} \circ \mathrm{Cores}^{\mathrm{G}_{xz}}_{\mathrm{G}_{xyz}} (h_1(x,y) \cup h_2(y,z)) \end{aligned}$$

And

$$\mathrm{Res}_{\underline{\mathrm{A}}}^{G_{xz}} \circ \mathrm{Cores}_{\mathrm{G}_{xyz}}^{\mathrm{G}_{xz}} (h_1(x,y) \cup h_2(y,z)) = \sum_{\underline{\mathrm{A}} \backslash \mathrm{G}_{xz}/\mathrm{G}_{xyz}} \mathrm{Cores}_{\underline{\mathrm{A}}_{y'}}^{\underline{\mathrm{A}}} \mathrm{Res}_{\underline{\mathrm{A}}_{y'}}^{\mathrm{G}_{xyz}} (h_1(x,y) \cup h_2(y,z)).$$

Lemma 3.10 shows that these $\mathrm{Cores}_{\underline{\mathrm{A}}_{y'}}^{\underline{\mathrm{A}}}(\ )$ vanish (under our hypothesis) unless $y' \in \mathrm{X}_*$ hence $\underline{\mathrm{A}}_{y'} = \underline{\mathrm{A}}$ and we get

$$\begin{aligned}(h_1 h_2)'(x,z) &= \sum_{y \in \mathrm{X}_*} \mathrm{Res}_{\underline{\mathrm{A}}}^{\mathrm{G}_{xyz}} (h_1(x,y) \cup h_2(y,z)) \\ &= \sum_{y \in \mathrm{X}_*} \mathrm{Res}_{\underline{\mathrm{A}}}^{\mathrm{G}_{xy}} h_1(x,y) \cup \mathrm{Res}_{\underline{\mathrm{A}}}^{\mathrm{G}_{yz}} h_2(y,z) \\ &= h_1' h_2'(x,z)\end{aligned}$$

□

4. Iwahori–Hecke Algebras

4.1. **Degree 0.** Define the *Iwahori–Hecke algebra* $\mathrm{H_I}$ in the same way as the spherical Hecke algebra $\mathrm{H_K}$, replacing K with I. As $q = 1 \mod \ell^r$, the structure of $\mathrm{H_I}$ is particularly simple, we have an isomorphism

$$\begin{aligned}\mathrm{H_I} &\simeq \mathrm{S}[\widetilde{\mathrm{W}}] \\ 1_{\mathrm{I}w\mathrm{I}} &\to w.\end{aligned}$$

This follows from the standard presentation of the Iwahori–Hecke algebra where relations $(T_s - q)(T_s + 1)$ simplify to $T_s^2 = 1$ in our case.

Put

$$\begin{aligned}\mathrm{H_{KI}} &= \{f : \mathrm{G} \to \mathrm{S} \text{ compact supported, left I-invariant, right K-invariant}\} \\ &\simeq \mathrm{Hom}_{\mathrm{S[G]}}(\mathrm{S[G/I]}, \mathrm{S[G/K]}) \\ \mathrm{H_{IK}} &= \{f : \mathrm{G} \to \mathrm{S} \text{ compact supported, left K-invariant, right I-invariant}\} \\ &\simeq \mathrm{Hom}_{\mathrm{S[G]}}(\mathrm{S[G/K]}, \mathrm{S[G/I]})\end{aligned}$$

Let V be any G-representation. The Hecke algebra $\mathrm{H_K}$ acts on the right on

$$\mathrm{V^K} = \mathrm{Hom}_{\mathrm{S[G]}}(\mathrm{S[G/K]}, \mathrm{V}),$$

by precomposition. Elements of $\mathrm{H_{KI}}$ induce homomorphisms $\mathrm{V^K} \to \mathrm{V^I}$, Venkatesh's slogan is $\mathrm{H_{KI}}$ *goes from* K*-invariants to* I*-invariants.*

Remark 9. Let $\mathrm{Z} = \mathrm{S}[\mathrm{X}_*]^{\mathrm{W}}$. Then every element of Z is central in $\mathrm{S}[\widetilde{\mathrm{W}}]$ so we have a map $\mathrm{Z} \to \mathrm{centre}(\mathrm{H_I})$, and all the $\mathrm{H_?}$ have the structure of Z-modules. In Lemma 4.5, it is shown that the bimodules $\otimes_{\mathrm{Z}} \mathrm{Z}_{\mathfrak{M}}$ induce inverse equivalences of categories between $\mathrm{H_K} \otimes_{\mathrm{Z}} \mathrm{Z}_{\mathfrak{M}}$ and $\mathrm{H_I} \otimes_{\mathrm{Z}} \mathrm{Z}_{\mathfrak{M}}$-modules where $\mathfrak{M}$ is a maximal ideal of Z over which the map $\mathrm{Z} \to \mathrm{S}[\mathrm{X}_*]$ is étale.

4.2. **Derived Iwahori–Hecke algebras.** We create derived versions in exactly the same way as before:

$$\mathscr{H}_{\mathrm{I}} = \mathscr{H}(\mathrm{G}, \mathrm{I}) = \mathrm{Ext}^*_{\mathrm{S[G]}}(\mathrm{S[G/I]}, \mathrm{S[G/I]}),$$

and this is isomorphic to an algebra of

$$h : (x, y) \in (\mathrm{G/I})^2 \rightsquigarrow h(x, y) \in \mathrm{H}^*(\mathrm{G}_{xy}, \mathrm{S})$$

with the same constraints (1)-(2) and product as before.

We also get derived versions $\mathscr{H}_{\mathrm{IK}}$ and $\mathscr{H}_{\mathrm{KI}}$ of H_{KI} and H_{IK}.

Lemma 10 (4.7)**.**

(1) The restriction map

$$\mathscr{H}_{\mathrm{I}} \to \{f : \widetilde{\mathrm{W}}^2 \to \mathrm{H}^*(\underline{\mathrm{A}}, \mathrm{S}) \text{ supported on finitely many } \widetilde{\mathrm{W}}\text{-orbits}\}$$

is an algebra homomorphism.

(2) We can restrict an element of $\mathscr{H}_{\mathrm{IK}}$ to a function $\widetilde{\mathrm{W}} \times \mathrm{X}_* \to \mathrm{H}^*(\underline{\mathrm{A}}, \mathrm{S})$ and this restriction is compatible with Part 1. Similarly for $\mathscr{H}_{\mathrm{KI}}$.

Proof uses some of the same ideas we have seen in the derived Satake proof.