DERIVED HECKE ALGEBRA IN THE TAYLOR-WILES SETTING, I

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1. Introduction

Let $\mathbf{G}$ be a simply connected semisimple split algebraic group over $\mathbf{Q}$. Fix a Borel $\mathbf{B} \subset \mathbf{G}$ and a maximal torus $\mathbf{A} \subset \mathbf{B}$. Fix a "base level" $K_0 \subset \mathbf{G}(\mathbf{A}_f)$ in the finite adeles, and let $Y(1) = Y(K_0)$ denote the usual adèlic double quotient.

Let π denote a tempered, cohomological cuspidal automorphic representation of $\mathbf{G}$ such that $\pi^{K_0} \neq 0$.

Let T denote the set of all places where π is ramified, or where K_0 is not hyperspecial. These are the "bad primes".

We have the derived Hecke algebra

$$\widetilde{\mathbf{T}} = \bigoplus_{v \notin T \cup \{p\}} \widetilde{\mathbf{T}}_v,$$

with $\mathbf{Z}_p$-coefficients, for some fixed prime $p \gg 0$.

Theorem 1.1 (7.6 in [1])**.** *Let $\mathfrak{m} \subset \mathbf{T}_{K_0}$ be the maximal ideal associated to π. Under some assumptions on $\mathfrak{m}$ (which we will describe), the cohomology group*

$$H^*(Y(1), \mathbf{Z}_p)_{\mathfrak{m}}$$

is generated as a $\widetilde{\mathbf{T}}$-module by $H^q(Y(1), \mathbf{Z}_p)_{\mathfrak{m}}$, where as usual

$$q = \frac{1}{2}(\dim Y(1) - \delta),$$

and $\delta = \operatorname{rank} \mathbf{G}(\mathbf{R}) - \operatorname{rank} K_\infty$ (where K_∞ is a maximal compact subgroup of $\mathbf{G}(\mathbf{R})$).

2. Notation and Assumptions

Let $\mathcal{G}/\mathbf{Z}_p$ be an integral model for $\mathbf{G}$, with Borel and maximal torus $\mathcal{A} \subset \mathcal{B}$. Let $A = T = \mathcal{A}_{\mathbf{F}_p}$ and $G = \mathcal{G}_{\mathbf{F}_p}$. Let $G^\vee$ denote the dual group of $\mathbf{G}$ over $\mathbf{Z}$, and let $B^\vee$ and $A^\vee = T^\vee$ denote the dual Borel and dual torus.

2.1. Assumptions on π.

For simplicity, we assume the coefficient field of π is $\mathbf{Q}$ and we write $\mathbf{T}_{K_0}$ for the usual Hecke algebra

$$\operatorname{image}(H_{K_0} \to \operatorname{End}_{D(\mathbf{Z}_p)}(\operatorname{Chains}(Y(1), \mathbf{Z}_p)))$$

where H_{K_0} is the $\mathbf{Z}_p$-module generated by all Hecke operators prime to the level and to p, and $\operatorname{Chains}(Y(1), \mathbf{Z}_p)$ denotes the complex of singular chains for $Y(1)$ in the derived category of $\mathbf{Z}_p$-modules. The representation π gives rise to a homomorphism $\mathbf{T}_{K_0} \to \mathbf{Z}_p \to k := \mathbf{F}_p$, whose kernel is the maximal ideal associated to (the mod p reduction of) π.

Choose $p \gg 0$ such that

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(1) $H^*(Y(1), \mathbf{Z}_p)$ is torsion-free, and we want $p \nmid (\#W)$, where W is the Weyl group of $\mathbf{G}$.

(2) There is a representation
$$\widetilde{\rho} : G_{\mathbf{Q}} \to G^\vee(\mathbf{T}_{K_0,\mathfrak{m}})$$
satisfying the "usual properties" (see Section 6.2 of [2] for the exact assumptions). Let $\overline{\rho} = \widetilde{\rho} \bmod p$.

(3) (No congruences) $\mathbf{T}_{K_0,\mathfrak{m}} \cong \mathbf{Z}_p$. In some sense this is saying that π is "minimally ramified".

(4) $H_j(Y(1), \mathbf{Z}_p)_{\mathfrak{m}}$ is nonzero only in degrees $[q, q+\delta]$.

2.2. Assumptions on $\widetilde{\rho}$ and $R_{\overline{\rho}}$.

- Assume $\overline{\rho}$ has "big image" (i.e. the image of $\overline{\rho}|_{\mathbf{Q}(\mu_{p^\infty})}$ contains the image of the k-points of the simply connected cover of $G^\vee$), which in particular implies that $\mathrm{End}_k(\overline{\rho}) = k^\times$.
- $\widetilde{\rho}$ is "crystalline at p", in a precise deformation theoretic sense as in Conjecture 6.1 in [1].
- $H^0(\mathbf{Q}_q, \mathrm{Ad}\overline{\rho}) = H^2(\mathbf{Q}_q, \mathrm{Ad}\overline{\rho}) = 0$ for all $q \in T \cup \{p\}$. This implies that the local deformation ring $R_{\overline{\rho}}$ is equal to $\mathbf{Z}_p$ if $q \in T$, and formally smooth if $q = p$.

3. Taylor-Wiles Primes

Definition 3.1. A Taylor-Wiles datum is a set $Q_n = \{q_1, \ldots, q_s\}$ of primes such that

(1) Q_n is disjoint from $T \cup \{p\}$.

(2) $p^n \mid (q_i - 1)$ for $i = 1, \ldots, s$.

(3) For $i = 1, \ldots, s$, $\overline{\rho}(\mathrm{Frob}_{q_i})$ is conjugate to a "strongly regular" element $\mathrm{Frob}_{q_i}^T \in T^\vee(k)$, which means that
$$\mathrm{Cent}_{G^\vee}(\mathrm{Frob}_{q_i}^T) = T^\vee.$$
Note there are $|W|$ choices of $\mathrm{Frob}_{q_i}^T$.

Briefly: these exist by the Chebotarev density theorem, and the fact that we have a big image assumption on $\overline{\rho}$.

4. Level Structures

For a Taylor-Wiles prime $q \in Q_n$, let $Y_1(q)$ denote the locally symmetric space whose level is the preimage of a unipotent radical under reduction mod q: $\mathcal{G}(\mathbf{Z}_q) \to \mathcal{G}(\mathbf{F}_q)$. We have a tower
$$Y_1(q) \to Y_1(q,n) \to Y_0(q),$$
where $Y_0(q)$ has Iwahori level, i.e. the preimage of $\mathcal{B}(\mathbf{F}_q)$ under the same reduction map.

This cover $Y_1(q) \to Y_0(q)$ is a Galois cover, with Galois group $\mathbf{A}(\mathbf{F}_q)$ and the second map is the unique subcover with Galois group
$$\mathbf{A}(\mathbf{F}_q)/p^n \cong (\mathbf{Z}/p^n\mathbf{Z})^r$$
where r is the rank of $\mathbf{A}$. In general, we set
$$Y_1^*(Q_n) = Y_1(q_1, n) \times_{Y(1)} \cdots \times_{Y(1)} Y_1(q_s, n)$$
and
$$Y_0(Q_n) = Y_0(q_1) \times_{Y(1)} \cdots \times_{Y(1)} Y_0(q_s)$$
Then
$$Y_1^*(Q_n) \to Y_0(Q_n)$$

is Galois with Galois group $T_n := \prod_{i=1}^{s} \mathbf{A}(\mathbf{F}_q)/p^n$ which is non-canonically isomorphic to $(\mathbf{Z}/p^n\mathbf{Z})^{rs}$. From now on, let $R = rs$, which will be the dimension of S_∞ once we carry out the patching process.

5. Local-Global Compatibility

5.1. **Relationship between $Y_0(Q_n)$ and $Y(1)$.** Let Q_n be a Taylor-Wiles datum. For $q \in Q_n$, let

$$H_{K_q}, H_{I_q}, H_{K_q,I_q}, H_{I_q,K_q}$$

be the (underived) spherical/Iwahori Hecke-algebras with $\mathbf{Z}/p^n\mathbf{Z}$-coefficients.

The centre of H_{I_q} can be identified with H_{K_q} and we assume that H_{K_q} acts on $H_*(Y_0(q), k)_\mathfrak{m}$ (via this identification) by means of the same (generalised) eigencharacter $H_{K_q} \to k$ by which H_{K_q} acts on π.

As a consequence of this, we find that

$$H^*(Y(1), \mathbf{Z}/p^n\mathbf{Z})_\mathfrak{m} \otimes_{(\bigotimes_{q\in Q_n} H_{K_q})} \bigotimes_{q\in Q_n} H_{K_q,I_q} \cong H^*(Y_0(Q_n), \mathbf{Z}/p^n\mathbf{Z})_\mathfrak{m}$$

and

$$H^*(Y_0(Q_n), \mathbf{Z}/p^n\mathbf{Z})_\mathfrak{m} \otimes_{(\bigotimes_{q\in Q_n} H_{K_q})} \bigotimes_{q\in Q_n} H_{I_q,K_q} \cong H^*(Y(1), \mathbf{Z}/p^n\mathbf{Z})_\mathfrak{m}$$

In particular, we have the decomposition

$$H^*(Y_0(Q_n), \mathbf{Z}/p^n\mathbf{Z})_\mathfrak{m} = \bigoplus_{\mathrm{Frob}^T} H^*(Y_0(Q_n), \mathbf{Z}/p^n\mathbf{Z})_{\mathfrak{m},\mathrm{Frob}^T}$$

where $\mathrm{Frob}^T = \{\mathrm{Frob}_{q_i}^T : i = 1, \dots, s\}$ and the subscript means that the "U_q Hecke operator" $I_q \chi I_q$ acts via multiplication by $\chi(\mathrm{Frob}_q^T)$, where $\chi \in X_*(T) \cong X^*(T^\vee)$ is a dominant cocharacter (with respect to the Borel $\mathbf{B}$). Note that

$$H^*(Y_0(Q_n), \mathbf{Z}/p^n\mathbf{Z})_{\mathfrak{m},\mathrm{Frob}^T} \cong H^*(Y(1), \mathbf{Z}/p^n\mathbf{Z})_\mathfrak{m}$$

5.2. **Relationship between $Y_0(Q_n)$ and $Y_1^*(Q_n)$.** Consider the universal deformation ring $R_{\overline{\rho},Q_n}^{\leq n}$ parametrizing deformations ρ of $\overline{\rho}$ that are

(1) unramified outside $Q_n \cup T \cup \{p\}$,
(2) crystalline at p
(3) inertia level $\leq n$ for all $q \in Q_n$, i.e. the action of tame inertia factors through I_q/p^n.

Consider the universal deformation $\sigma : G_\mathbf{Q} \to G^\vee(R_{\overline{\rho},Q_n}^{\leq n})$.

Lemma 5.1 (6.12 in GV). *If Q_n is a Taylor-Wiles set and $q \in Q_n$, then $\sigma_{G_{\mathbf{Q}_p}}$ can be uniquely conjugated to a representation*

$$G_{\mathbf{Q}_q} \to T^\vee(R_{\overline{\rho},Q_n}^{\leq n}).$$

landing in the torus where the image of a fixed uniformizer is $\mathrm{Frob}_{q_i}^T$.

If we restrict to $\mathbf{F}_q^\times \subset \mathbf{Q}_q^\times$, then we have

$$\mathbf{F}_q^\times \to \mathbf{F}_q^\times/p^n \to T^\vee(R_{\overline{\rho},Q_n}^{\leq n})$$

and by pairing with characters in $X^*(T^\vee)$, we get a map

$$A(\mathbf{F}_q) \to \mathbf{A}(\mathbf{F}_q)/p^n \to (R_{\overline{\rho},Q_n}^{\leq n})^\times$$

Then we have a map

$$T_n \to (R_{\overline{\rho},Q_n}^{\leq n})^\times \to \mathbf{T}_\mathfrak{m}.$$

Assume the action of T_n on $H^*(Y_1^*(Q_n), \mathbf{Z}/p^n\mathbf{Z})_{\mathfrak{m},\mathrm{Frob}^T}$ via the above map coincides with the action via deck transformations, where T_n is the Galois group of $Y_1^*(Q_n)$ over $Y_0(Q_n)$.

6. Patching

Let R denote the dimension of the dual Selmer group for $\mathrm{Ad}\overline{\rho}$ associated to the deformation functor $\mathrm{Def}_{\overline{\rho}}^{\mathrm{cris}}$.

(1) Define the group ring $S_n = \mathbf{Z}/p^n\mathbf{Z}[T_n]$, and let I_n denote the augmentation ideal, so that $S_n/I_n \cong \mathbf{Z}/p^n\mathbf{Z}$. Let

$$S_\infty = \mathbf{Z}_p\|x_1, \ldots, x_R\|$$

and I_∞ denote the ideal $(x_1, \ldots, x_R)$.

(2) We have perfect complexes

$$C_0 = \mathrm{Chains}(Y(1), \mathbf{Z}_p)_{\mathfrak{m}}$$

and

$$C_n = \mathrm{Chains}(Y_1^*(Q_n), \mathbf{Z}_p)_{\mathfrak{m},\mathrm{Frob}^T}$$

(3) We have deformation rings

$$R_0 = R_{\overline{\rho}} \to \mathrm{End}_{D(\mathbf{Z}_p)}(C_0)$$

and

$$R_n = R^{\leq n}_{\overline{\rho},Q_n}/(p^n, \mathfrak{m}^{k(n)}).$$

There is a map $S_n \to R_n$, and can assume $k(n) \geq 2n$.

(4) By the formal smoothness assumption on $\overline{\rho}$, we get surjective maps

$$R_\infty = \mathbf{Z}_p\|x_1, \ldots, x_{R-\delta}\| \twoheadrightarrow R_n$$

such that $C_n/I_n \cong C_0/p^n$: this follows from local-global compatibility. We have a diagram

$$\begin{array}{ccc} R_n & \longrightarrow & \mathrm{End}_{D(S_n)}(C_n) \\ \downarrow & & \downarrow \\ R_n/p^n & \longrightarrow & \mathrm{End}_{D(\mathbf{Z}/p^n)}(C_0/p^n) \end{array}$$

We can choose a sequence of Taylor-Wiles data $\{Q_n\}_{n\geq 1}$. Then

(1) we get a perfect complex C_∞ of S_∞-modules concentrated in $[-(q+\delta), -q]$, such that

$$C_\infty \otimes^{\mathbf{L}}_{S_\infty} S_n \cong C_n$$

and

$$C_\infty \otimes^{\mathbf{L}}_{S_\infty} \mathbf{Z}_p \cong C_0$$

(2) The map $S_\infty \to R_\infty$ is surjective by the no congruences assumption.

(3) Moreover, C_∞ is quasi-isomorphic to to $H_q(C_\infty)$ with the latter free over R_∞.

7. Proof of Theorem 1

By definition of the derived Hecke algebra, it is enough to prove it for $H^*(Y(1), \mathbf{Z}/p^n\mathbf{Z})_{\mathfrak{m}}$ for all $n \geq 1$.

Assume that

$$H^q(\mathrm{Hom}_{S_\infty}(C_\infty, \mathbf{Z}_p)) \times \mathrm{Ext}^j_{S_\infty}(\mathbf{Z}_p, \mathbf{Z}_p) \twoheadrightarrow H^{q+j}(\mathrm{Hom}_{S_\infty}(C_\infty, \mathbf{Z}_p))$$

is a surjection.[1] Note

$$H^q(\mathrm{Hom}_{S_\infty}(C_\infty, \mathbf{Z}_p)) = \mathrm{Hom}_{D(S_\infty)}(C_\infty, \mathbf{Z}_p[q])$$

and

$$\mathrm{Ext}^j_{S_\infty}(\mathbf{Z}_p, \mathbf{Z}_p) = \mathrm{Hom}_{D(S_\infty)}(\mathbf{Z}_p[q], \mathbf{Z}_p[q+j]),$$

so the map is just the natural composition map coming from this description.

We pass to level n:

$$\begin{array}{ccccc}
H^q(\mathrm{Hom}_{S_\infty}(C_\infty, \mathbf{Z}_p)) & \times & \mathrm{Ext}^j_{S_\infty}(\mathbf{Z}_p, \mathbf{Z}_p) & \longrightarrow & H^{q+j}(\mathrm{Hom}_{S_\infty}(C_\infty, \mathbf{Z}_p)) \\
\| & & \downarrow & & \downarrow\!\!\downarrow (\dagger) \\
H^q(\mathrm{Hom}_{S_\infty}(C_\infty, \mathbf{Z}_p)) & \times & \mathrm{Ext}^j_{S_\infty}(\mathbf{Z}_p, \mathbf{Z}/p^n\mathbf{Z}) & \longrightarrow & H^{q+j}(\mathrm{Hom}_{S_\infty}(C_\infty, \mathbf{Z}/p^n\mathbf{Z})) \\
\downarrow (**) & & \uparrow\!\!\uparrow (*) & & \uparrow \sim \\
H^q(\mathrm{Hom}_{S_n}(C_n, \mathbf{Z}/p^n\mathbf{Z}) & \times & \mathrm{Ext}^j_{S_n}(\mathbf{Z}/p^n\mathbf{Z}, \mathbf{Z}/p^n\mathbf{Z}) & \longrightarrow & H^{q+j}(\mathrm{Hom}_{S_n}(C_n, \mathbf{Z}/p^n\mathbf{Z}))
\end{array}$$

This diagram is commutative in the sense that the maps $(*)$ and $(**)$ are adjoint. Furthermore, $(*)$ is surjective by the unnumbered Lemma in [2, §6.4] and $(\dagger)$ is surjective by the no torsion assumption. Tracing through the diagram, we see that the bottom row is surjective. But

- $H^{q+j}(\mathrm{Hom}_{S_n}(C_n, \mathbf{Z}/p^n\mathbf{Z})) \cong H^{q+j}(Y_0(Q_n), \mathbf{Z}/p^n\mathbf{Z})_{\mathfrak{m},\mathrm{Frob}^T}$
- $\mathrm{Ext}^j_{S_n}(\mathbf{Z}/p^n\mathbf{Z}, \mathbf{Z}/p^n\mathbf{Z}) = H^j(T_n, \mathbf{Z}/p^n\mathbf{Z})$ and the action factors through

  $$\bigotimes_{q \in Q_n} \mathcal{H}_{I_q}$$

  by pulling back under $Y_0(Q_n) \to BT_n$ and cupping.

We have three *surjective* maps

(1)
$$H^q(Y(1), \mathbf{Z}/p^n\mathbf{Z})_{\mathfrak{m}} \otimes_{(\bigotimes_{q \in Q_n} H_{K_q})} \bigotimes_{q \in Q_n} H_{K_q, I_q} \twoheadrightarrow H^q(Y_0(Q_n), \mathbf{Z}/p^n\mathbf{Z})_{\mathfrak{m}}$$

(2)
$$H^q(Y_0(Q_n), \mathbf{Z}/p^n\mathbf{Z})_{\mathfrak{m}} \otimes_{(\bigotimes_{q \in Q_n} H_{K_q})} \bigotimes_{q \in Q_n} \mathcal{H}_{I_q} \twoheadrightarrow H^q(Y_0(Q_n), \mathbf{Z}/p^n\mathbf{Z})_{\mathfrak{m}}$$

(3)
$$H^{q+j}(Y_0(Q_n), \mathbf{Z}/p^n\mathbf{Z})_{\mathfrak{m}} \otimes_{(\bigotimes_{q \in Q_n} H_{K_q})} \bigotimes_{q \in Q_n} H_{I_q, K_q} \twoheadrightarrow H^{q+j}(Y(1), \mathbf{Z}/p^n\mathbf{Z})_{\mathfrak{m}}$$

[1]This is proven in [2, Appendix B] and involves a Koszul resolution calculation. The key point is that C_∞ is quasi-isomorphic to a free R_∞-module, and R_∞ is a quotient of S_∞.

where the first and third maps follow from local-global compatibility. Furthermore the actions of these Hecke algebras are compatible under the morphisms $H_{K_q,I_q} \otimes \mathcal{H}_{I_q} \otimes H_{I_q,K_q} \to \mathcal{H}_{K_q}$.

Thus $\bigotimes_q \mathcal{H}_{K_q}$ acts surjectively on $H^*(Y(1), \mathbf{Z}/p^n\mathbf{Z})_{\mathfrak{m}}$.

References

[1] S. Galatius and A. Venkatesh. Derived Galois deformation rings. *Adv. Math.*, 327:470–623, 2018.

[2] Akshay Venkatesh. Derived Hecke algebra and cohomology of arithmetic groups. *arXiv e-prints*, page arXiv:1608.07234, Aug 2016.